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Chapetr 1 :Vector Spase

Definition *1.1.1*

Real vector spases :

A real vector space V is a set of objects,called vectors together with two operations called
addition and scalar multiplication that satisfy the ten axioms listed below:

i) If $x \in V$ and $y \in V$ then $x + y \in V$ (closure under addition).

ii) For all x, y and z in V

$(x + y) + z = x + (y + z)$ (associative law of vector addition)

iii)|There is vector $0 \in V$ s.t for all $x \in V$

$x + 0 = 0 + x = x$ (0 is the additive identity)

iv) If $x \in V \exists - x \in V$ s.t

$x + (-x) = (-x) + x = 0$ ($-x$ is the additive inverse of x)

v) If x and $y \in V$ then

$x + y = y + x$ (commutative law of vector addition)

vi) If $x \in V$ and α is a scalar,

then $\alpha x \in V$ (colsure under scalar multiplicution)

vii) If $x, y \in V$ and α scalar,

then $\alpha(x + y) = \alpha x + \alpha y$ (first distributive law)

viii) If $x \in V, \alpha, \beta$ scalars ,

then $(\alpha + \beta)x = \alpha x + \beta x$ (second distributive law)

ix) If $x \in V$ and α, β scalars ,

then $\alpha(\beta x) = (\alpha\beta)x$ (associative law of scalar multiplication)

x) For every vector $x \epsilon V, 1x = x$ (the scalar 1 is called
a multipticalive identity)

Example *(1):*

let $V = R^n = \{(x_1, x_2, \ldots\ldots, x_n) : x_i \in R,$ for $i = 1, 2, \ldots, n\}$

$V = R^n$ satisfy all axioms of a vector space.

Example *(2):*

let $V = P_n$ the set of all polynomials with real coefficients of degree less than or equal to n if

$p \in P_n$,then

$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots\ldots + a_1 x + a_0$

$q(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots\ldots\ldots + b_1 x + b_0$

$p(x) + q(x) = (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots\ldots + (a_1 + b_1)x + (a_0 + b_0)$

$0 = 0x^n + 0x^{n-1} + 0x^{n-2} + \ldots\ldots + 0x + 0$

$-p(x) = -a_n x^n - a_{n-1} x^{m-1} - \ldots\ldots\ldots - a_1 x - a_0$

$\alpha p(x) = \alpha a_n x^n + \alpha a_{n-1} x^{n-1} + \ldots\ldots\ldots + \alpha a_1 x + \alpha a_0$

Theorem *1.1.2*

let V be a vector space (0 is zero vector)(x any vector), then
i) $\alpha 0 = \underline{0}$ for every real number α
ii) $0\underline{x} = \underline{0}$ for every $x \in V$
iii) If $\alpha\underline{x} = \underline{0}$ then $\alpha = 0$ or $\underline{x} = \underline{0}$ (or both)
iv) $(-1)\underline{x} = -\underline{x}$ for every $x \in V$

1.2 : Sub spaces

Definition *1.2.1*

let H be a nonempty subset of a vector spase V and suppose that H is it self a vector

space
under the opertions of addition and scalar multiplication defined on V.
Then H is said to be a subspase of V.

Theorem *1.2.2*

A nonempty subset H of the vector spase V is a subspace of V
if the two closure rules hold:
i) If $x \in H$ and $y \in H$, Then $x + y \in H$
ii) If $x \in H$, Then $\alpha x \in H$ for every scalur α .

Every vector spase V contain two proper subspace $\{0\}$ and V.

Example *(3)*

let $V = \{(x, y) : x, y \in R\} = R^2$
$W = \{(x, 2x) : x \in R\}$ is a supspace of R^2. prove?

1.3: Linear Dependence and Independence :

Definition *1.3.1*

* let $v_1, v_2, \ldots, v_n$ be n vectors in a vector space V.
Then the vector are said to be linearly dependent if there
exist n scalars $c_1, c_2, \ldots\ldots, c_n$ not all zero such that

$$c_1 v_1 + c_2 v_2 + \ldots\ldots + c_n v_n = 0$$

not all $c_i = 0$.
* If the vectors are not linearly dependent, they are
linearly independent , if

$$c_1 v_1 + c_2 v_2 + \ldots\ldots + c_n v_n = 0$$

it holds for $c_1 = c_2 = \ldots\ldots = c_n = 0$.

1.4 : Basis and Dimension :

A set of vector $\{v_1, v_2, \ldots\ldots, v_n\}$ form a basis for V if :
i) $\{v_1, v_2, \ldots\ldots, v_n\}$ is linearly independent .
ii) $\{v_1, v_2, \ldots\ldots, v_n\}$ spans V.

Definition *1.4.1:*

If the vector space V has a finite basis , then the dimension of V is the number of vector in the basis
and V is called a finit dimensional vector space.otherwise V is called an infinite dimensional vector space .
If $V = \{0\}$, then V is siaid to be zero dimensional .

1.5 : Coordinates and change of Basis :

Definition *1.5.1:*

let $B = \{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for a vector space V and x a vector in V such that

$$x = c_1 v_1 + c_2 v_2 + \ldots + c_n v_n$$

Then the scalars $c_1, c_2, \ldots\ldots, c_n$ are called the coordinate of x relative to the basis B
The coordinate of x relative to B is the vector in R^n denoted by

$$(x)_B = [x]_B = (c_1, c_2, \ldots\ldots, c_n).$$

Example *(4)*

Find the coordinate vector of $x = (-2,1,3)$ in R^3 relative to the standrad basis
$S = \{(1,0,0),(0,1,0),(0,0,1)\}$
$x = (-2,1,3) = c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1)$
$= -2(1,0,0) + 1(0,1,0) + 3(0,0,1)$
$(x)_B = (-2,1,3).$

Example *(5)*

Find the vector x in R^2 relative to the nonstandrad basis $B = \{(1,0),(1,2)\}$ where $(x)_B = (3,2)$ and Find the coordinate vector of x relative to the standrad basis
$B' = \{(1,0),(0,1)\}$
$(x)_B = (3,2)$
$x = 3(1,0) + 2(1,2) = (5,4)$
$(5,4) = 5(1,0) + 4(0,1)$
so $(x)_{B'} = (5,4).$

1.5.2 Change of basis in R^n

The procedure demonstrated in EX(5) is called change of basis That is we were given the coordinat of
a vector relative to are basis B and asked to finite the coordinates relative to another basis B'.

In this case we need to know the following :

P is the transition matrix from B' to B

$[x]_{B'}$ is the coordinate matrix of x relative to B'.

$[x]_B$ is the coordinate matrix of x relative to B.

Multiplication by the transition matrix P changes the coordinate matrix relative to B' into a coordinate matrix relative to B.

That is

$$P[x]_{B'} = [x]_B$$

change of basis from B' to B

To preform a change of basis from B to B', we use the matrix P^{-1} (the transition matrix from B to B')

$$[x]_{B'} = P^{-1}[x]_B$$

change of basis from B to B'.

Theorem *1.5.3:*

If P is the transition matrix from a basis B' to a basis B in R^n, then P is invertible and the transition matrix from B to B' is given by P^{-1}.

Theorem *1.5.4:*

let $B = \{v_1, v_2, \ldots\ldots, v_n\}$ and $B' = \{u_1, u_2, \ldots\ldots, u_n\}$ be two basis for R^n

then the transition matrix P^{-1} from B to B' can be found by using Gauss-Jordan elimination on the $n \times 2n$ matrix $[B' \vdots B]$ as following

$$[B' \mid B] \Rightarrow [I_n \mid P^{-1}]$$

$$[B \mid B'] \Rightarrow [I_n \mid P]$$

Example *(6):*

Find the transition matrix from B to B' for the following bases in R^3

$B = \{(1,0,0),(0,1,0),(0,0,1)\}$ and $B' = \{(1,0,1),(0,-1,2),(2,3,-5)\}$

Solution *:*

We want to find P^{-1}?

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{pmatrix}$$

$$[B' \vdots B] = \begin{pmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & -1 & 3 & \vdots & 0 & 1 & 0 \\ 1 & 2 & -5 & \vdots & 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -3 & \vdots & 0 & -1 & 0 \\ 0 & 2 & -7 & \vdots & -1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 2 & \vdots & 1 & 0 & 0 \\ 0 & 1 & -3 & \vdots & 0 & -1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & -2 & -1 \end{pmatrix} \rightharpoonup \begin{pmatrix} 1 & 0 & 0 & \vdots & -1 & 4 & 2 \\ 0 & 1 & 0 & \vdots & 3 & -7 & -3 \\ 0 & 0 & 1 & \vdots & 1 & -2 & -1 \end{pmatrix}$$

$$\therefore P^{-1} = \begin{pmatrix} -1 & 4 & 2 \\ 3 & -7 & -3 \\ 1 & -2 & -1 \end{pmatrix}$$

Note:

a) When B is the standrad basis
$[B' \mid B]$ to $[I_n \mid P^{-1}]$ becomes

$$[B' \mid I_n] \Rightarrow [I_n \mid P^{-1}]$$

i.e
$(B')^{-1} = P^{-1}$ standrad basis to nonstandrad basis

b) When B' is the standrad basis
$[B' \mid B]$ to $[I_n \mid P^{-1}]$ becomes

$$[I_n \mid B] \Rightarrow [I_n \mid P^{-1}]$$

i.e
$P^{-1} = B$ nonstandrad basis to standrad basis.

1.5.5 coordinate Representation in general n-Dimensional spaces

Example *(7):*

Find the Coordinate vector of $p = 3x^3 - 2x^2 + 4$ relative to the standrad basis of $P_3, S = \{1, x, x^2, x^3\}$

$p = 4(1) + 0(x) + (-2)(x^2) + 3(x^3)$

$(p)_s = (4, 0, -2, 3)$

Example *(8):*

Find the Coordinate vector of $x = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix}$ relative to the standrad basis of $M_{3,1}$

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$x = \begin{pmatrix} -1 \\ 4 \\ 3 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$(x)_S = (-1, 4, 3).$

Example *(9):*

Consider the basis $B = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}, B' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

a) Find the transition matrix from B to B'.

b)Find $[v]_{B'}$ if $v = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$.

Solution *:*

$u_1 = au_1' + bu_2'$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} + \begin{pmatrix} 2b \\ b \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a+2b \\ a+b \end{pmatrix}$$

$b = 1, a = -1$

$$[u_1]_{B'} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$u_2 = cu_1' + du_2'$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c \\ c \end{pmatrix} + \begin{pmatrix} 2d \\ d \end{pmatrix}$$

$d = -1, c = 2$

$$[u_2] = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$\therefore\ P^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

To find $[v]_{B'}$ we need to find $[v]_B$

$$\begin{pmatrix} 7 \\ 2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\Rightarrow\ a = 7, b = 2$

$$[v]_B = \begin{pmatrix} 7 \\ 2 \end{pmatrix}$$

$$[v]_{B'} = p^{-1}[v]_B$$

$$= \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 7 \\ 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$$

if we want to find P we

$u_1' = au_1 + bu_2$

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow a = 1, b = 1$$

$$[u_1'] = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$u_2' = cu_1 + du_2$

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = c\begin{pmatrix} 1 \\ 0 \end{pmatrix} + d\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow c = 2, d = 1$$

$$[u_2']_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\therefore\ P = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$$

$$P^{-1}P = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

Example *(10):*

let $B = \left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\}$ and $B' = \left\{\begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \end{pmatrix}\right\}$ be two basis of R^2

if $[x]_B = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$

write x in terms of the vectors in B'.

Solution *:*

$$\begin{pmatrix} 3 \\ 1 \end{pmatrix} = a\begin{pmatrix} 2 \\ 4 \end{pmatrix} + b\begin{pmatrix} -5 \\ 3 \end{pmatrix} \Rightarrow a = \tfrac{7}{13}, b = \tfrac{-5}{13}$$

$$[u_1]_{B'} = \begin{pmatrix} \frac{7}{13} \\ \frac{-5}{13} \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ -1 \end{pmatrix} = c\begin{pmatrix} 2 \\ 4 \end{pmatrix} + d\begin{pmatrix} -5 \\ 3 \end{pmatrix} \Rightarrow c = \frac{1}{26}, d = \frac{-5}{26}$$

$$[u_2]_{B'} = \begin{pmatrix} \frac{1}{26} \\ \frac{-5}{26} \end{pmatrix}$$

$$\therefore\ P^{-1} = \begin{pmatrix} \frac{7}{13} & \frac{1}{26} \\ \frac{-5}{13} & \frac{-5}{13} \end{pmatrix}$$

$$[x]_{B'} = P^{-1}[x]_B = \begin{pmatrix} \frac{7}{13} & \frac{1}{26} \\ \frac{-5}{13} & \frac{-5}{13} \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix} = \frac{1}{26}\begin{pmatrix} 14 & 1 \\ -10 & -10 \end{pmatrix}\begin{pmatrix} 7 \\ 4 \end{pmatrix}$$

$$= \frac{1}{26}\begin{pmatrix} 102 \\ -110 \end{pmatrix} = \begin{pmatrix} \frac{51}{13} \\ \frac{-55}{13} \end{pmatrix}$$

$$x = \frac{51}{13}(2,4) - \frac{55}{13}(-5,3) = \left(\frac{377}{13}, \frac{39}{13}\right).$$

1.6 :Applications of vector spaces :

1.6.1 : Linear differential equations

A linear differential equation of order n is of the form

$$y^{(n)} + g_{n-1}(x)y^{(n-1)} + \ldots + g_1(x)y^1 + g_\circ(x)y = f(x).$$

where $g_1, g_2, \ldots, g_n$ and f are functions of x with a common domain .

if $f(x) = 0$ the equation is homogeneous.
otherwise its nonhomogeneous.

Afunction y is a solution of the linear differential equation if the equation is satisfied

when y
and its first n derivatives are substituted into the equation.

Example *(11):*

Show that both $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the second order linear differential equation

$y'' - y = 0$

Solution *:*

For the function $y_1 = e^x$ we have $y_1' = e^x, y_1'' = e^x$

$y_1'' - y_1 = e^x - e^x = 0$

so y_1 is a solution of the S.O.L.D.equation

For $y_2 = e^{-x}$ $\quad y_2' = -e^{-x}, y_2'' = e^{-x}$

$y_2'' - y_2 = e^{-x} - e^{-x} = 0$

so y_2 is asolution of the given L.D. equation.

From last example we see that in the vector space $C''(-\infty, \infty)$ of all twice differentiable

function defined on the entire real line .

the two sol. $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent . This mean that the only sol. of

$c_1 y_1 + c_2 y_2 = 0$ is valid for all $c_1 = c_2 = 0$

Also every linear combination of y_1 and y_2 is also a solution of the given L.D.eq.

let $y = c_1 y_1 + c_2 y_2$ then

$y = c_1 e^x + c_2 e^{-x}$

$y' = c_1 e^x - c_2 e^{-x}$

$y'' = c_1 e^x + c_2 e^{-x}$

substituting into $y'' - y = 0$

$y'' - y = (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) = 0$

Thus $y = c_1 e^x + c_2 e^{-x}$ is a solution.

1.6.2 :**Solution of a linear Homogeneous Differential equation** :

Every n[th] order linear homogenous differential equation

$$y^{(n)} + g_{n-1}(x) y^{(n-1)} + \ldots + g_1(x) y' + g_\circ(x) y = 0$$

has n linearly independent solutions , moreover , if $\{y_1, y_2, \ldots, y_n\}$ is a set of L.I.N. solution ,

then every solution is of the form

$$y = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n \qquad *$$

where $c_1, c_2, \ldots, c_n$ are real numbers.

we call * the general solution.

1.6.3 : **Definition of the Wronskian of a set of functions** :

let $\{y_1, y_2, \ldots, y_n\}$be a set of functions each of which possesses n-1 derivatives on an interval I. The determinant

$$w(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \vdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

is called the wronskian of the given set of functions.

Remark *:*

The wronskian of a set of functions is named after the mathimatician Josef Maria Wronski.

Example *(12):*

Find the wronskian of a set of functions.

a) $\{1-x, 1+x, 2-x\}$ is

$$w = \begin{vmatrix} 1-x & 1+x & 2-x \\ -1 & 1 & -1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

b) $\{x, x^2, x^3\}$

$$w = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x(12x^2 - 6x^2) - (6x^3 - 2x^3) = 6x^3 - 4x^3 = 2x^3$$

The wronskian in (a) is said to be identically equal to zero because its zero for any value of x.

The wronskian in (b) is not identically equal to zero because values of x exist for which this wronskian is nonzeros.

Theorem *(1.6.4)* ***Wronskian Test for linear independence****:*

Let $\{y_1, y_2, \ldots\ldots, y_n\}$ be a set of n solution of an n^{th} order linear homogeneous differential equation

this set is linearly independent iff the wronskian is not identically equal to zero.

Example *(13):*

Determine whether $\{1, \cos x, \sin x\}$ is a set of linearly independent solution of the linear homogeneous
differential equation $y''' + y' = 0.$

Solution *:*

$y_1 = 1$ $\qquad y_2 = \cos x$ $\qquad y_3 = \sin x$
$y_1' = 0$ $\qquad y_2' = -\sin x$ $\qquad y_3' = \cos x$
$y_1'' = 0$ $\qquad y_2'' = -\cos x$ $\qquad y_3'' = -\sin x$
$y_1''' = 0$ $\qquad y_2''' = \sin x$ $\qquad y_3''' = -\cos x$

for y_1 we get
$y''' + y' = 0 + 0 = 0$
for y_2 we get
$y''' + y' = \sin x - \sin x = 0$
for y_3 we get
$y''' + y' = -\cos x + \cos x = 0$
so $\{1, \cos x, \sin x\}$ is a solution of the H.D.L.equation.
Now we test for L.I.N

$$w = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1$$

so w is not identically equal to zero.we conclude the set $\{1, \cos x, \sin x\}$ is L.I.N.
since the set consists of 3 L.I.N ;solutions of a thired order linear homogeneous differential equation
we conclude that the general solution is
$y = c_1 + c_2 \cos x + c_3 \sin x.$

PROBLEM SET I

(1) Find the length of the given vector

(i) $v = (4,3)$

(ii) $v = (1,0,0)$

(iii) $v = (4,0,-3,5)$

(2) Find (a) $\|u\|$, (b) $\|v\|$, and (c) $\|u+v\|$

(i) $u = (1, \frac{1}{2}), v = (2, \frac{-1}{2})$

(ii) $u = (0,1,-1,2), v = (1,1,3,0)$

(3) Find a unit vector (a) in the direction of u and (b) in the direction opposite that of u.

(i) $u = (3,2,-5)$

(ii) $u = (1,0,2,2)$

(4) For what values of c is $\|c(1,2,3)\| = 1$?

(5) Find the vector v with the given length that has the same direction as the vector u.

$\|v\| = 2, \qquad u = (\sqrt{3},3,0).$

(6) Given the vector $v = (8,8,6)$, find u such that

(a) u has the same direction as v and one-half its length.

(b) u has the direction opposite that of v and one-fourth its length.

(7) Find the distance between u and v

(i) $u = (3,4), v = (7,1)$

(ii) $u = (1,1,2), v = (-1,3,0)$

(8) Find (a) $u \cdot v$, (b) $u \bullet u$, (c) $\|u\|^2$,

(d) $(u \bullet v)v$, and (e) $u \bullet (2v)$

(i) $u = (3,4), v = (2,-3)$

(ii) $u = (2,-3,4), v = (0,6,5)$

(iii) $u = (4,0,-3,5), v = (0,2,5,4)$

(9) Find $(u+v) \bullet (2u-v)$, given that

$u \bullet u = 4, \qquad u \bullet v = -5,$ and $v \bullet v = 10.$

(10) Verify the Cauchy-Schwarz Inequality for the given vectors.

$u = (3,4) \qquad v = (2,-3)$

(11) Find the angel θ between the given vectors

(i) $u = (1,1), v = (2,-2)$

(ii) $u = (1,1,1), v = (2,1,-1)$.

(12) Determine all vectors v that are orthogonal to the given vector u.

(i) $u = (0,5)$

(ii) $v = (4,-1,0)$

(13) Determine wether u and v are orthogonal ,parallel,or nither.

(i) $u = (4,0), v = (1,1)$

(ii) $u = (0,1,6), v = (1,-2,-1)$

(14) Verify the Triangle Inequality for the given vectors.

$u = (4,0), v = (1,1)$

(15) Verify the pythagorean Theorem for the given vectors.

$u = (1,-1), v = (1,1)$

(16) Prove that if u is orthogonal to v and w, then u is orthogonal to $cv + dw$ for any scalars c and d.

(17) You are given the coordinate vector of x relative to a nonstandrad basis B. Find the coordinate vector of x relative to the standrad basis in R^n.

(a) $B = \{(2,-1),(0,1)\}$, $[x]_B = (4,1)$

(b) $B = \{(1,0,1),(1,1,0),(0,1,1)\}$, $[x]_B = (2,3,1)$

(18) Find the transition matrix from B to $B^\backslash$

(a) $B = \{(2,4),(-1,3)\}$, $B^\backslash = \{(1,0),(0,1)\}$

(b) $B = \{(1,0,2),(0,1,3),(1,1,1)\}$, $B^\backslash = \{(2,1,1),(1,0,0),(0,2,1)\}$

(19) (a) In Ex (18) (a) find $[x]_B$ given $[x]_{B^\backslash} = (-1,3)$

(b) In Ex (18) (b) find $[x]_B$ given $[x]_{B^\backslash} = (1,2,-1)$

(20)Find the coordinate vector of p relative to the standrad bassis in P_2

$p = x^2 + 11x + 4$.

Problem set II

(1)Find the wronskian for the given set of functions

(a) $\{e^{x}, e^{-x}\}$ (b) $\{1, e^{x}, e^{2x}\}$ (c) $\{x, \sin x, \cos x\}$

(2)Test the given set of solutions for linear independence and find the general solution

(a) $y'' + y = 0,$ solution $\{\sin x, \cos x\}$

(b) $y''' + 4y'' + 4y' = 0,$ solution $\{e^{-2x}, xe^{-2x}, (2x+1)e^{-2x}\}.$

ChapterII

Inner Product Spaces

2.1 Lenght and Dat product in R^n

In Ch.I its mentioned that vectors in the plane can be defined by a directed line segments having
a certain length and direction . In this section we use R^n as a model is define these and other geometric
properties (such as distance and angle) for vectors in R^n .

Definition *2.1.1:*

The length of a vector $v = (v_1, v_2, \ldots\ldots, v_n)$ in R^n is given by

$$\|v\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$$

Remark *2.1.2:*

- The length of a vector is called its norm.
- If $\|v\| = 1$ then the vector $\underline{v}$ is called the unit vector.
- $\|v\| \geq 0$, $\|v\| = 0$ iff $\underline{v}$ is the zero vector.

Each vector in the standard basis for R^n has length 1 and is called the standard unit vector in R^n.

we denote the standred unit vectors in R^2 and R^3 as follows:

$$\{i, j\} = \{(1,0), (0,1)\}$$
$$\{i, j, k\} = \{(1,0,0), (0,1,0), (0,0,1)\}$$

Example *(1):*

a) Find the length of the vector $\underline{v} = (0, -2, 1, 4, -2)$ in R^5

$$\|\underline{v}\| = \sqrt{0 + (-2)^2 + (1)^2 + (4)^2 + (-2)^2}$$

$$= \sqrt{0 + 4 + 1 + 16 + 4} = \sqrt{25} = 5$$

b) Find the length of the vector $\underline{v} = \left(\frac{2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ in R^3

$$\|\underline{v}\| = \sqrt{\left(\frac{2}{\sqrt{17}}\right)^2 + \left(\frac{-2}{\sqrt{17}}\right)^2 + \left(\frac{3}{\sqrt{17}}\right)^2} = \sqrt{\frac{4}{17} + \frac{4}{17} + \frac{9}{17}} = \sqrt{\frac{17}{17}} = 1$$

v is a unit vector.

Note:

Two nonzero vectors u and v in R^n are parallel if one is a scalar multiple of the other
i.e, $u = cv$

a) If $c > 0$ then u, v have the same direction.

b) if $c < 0$ then u, v have opposite direction.

Theorem *2.1.3:*

Let v be a vector in R^n and c a scolar, then

$$\| cv \| = |c| \| v \|,$$

where $|c|$ is the absolut value of c.

Proof :

$cv = (cv_1, cv_2, \ldots\ldots, cv_n)$

$$\| cv \| = \sqrt{(cv_1)^2 + (cv_2)^2 \ldots\ldots\ldots + (cv_n)^2}$$

$$= \sqrt{c^2(v_1^2 + v_2^2 \ldots\ldots + v_n^2)}$$

$$= |c| \sqrt{v_1^2 + v_2^2 \ldots\ldots\ldots + v_n^2}$$

$$= |c| \; \| v \|.$$

Theorem *2.1.4:*

If v is a nonzero vector in R^n , then the vector $u = \frac{v}{\| v \|}$

has length L and has the same direction as v. we call u , the unit vector in the direction of v.

Proof :

Since v is nonzero so $\| v \| \neq 0$

Thus $\frac{1}{\| v \|}$ is positive

let u be a positive sclaler multiple of v

$$u = \frac{1}{\| v \|} v$$

$$u = \frac{(v_1, v_2, \ldots\ldots, v_n)}{\sqrt{v_1^2 + v_2^2 + \ldots\ldots\ldots\ldots + v_n^2}}$$

$$\| u \| = \sqrt{\frac{v_1^2}{v_1^2 + v_2^2 + \ldots\ldots + v_n^2} + \frac{v_2^2}{v_1^2 + v_2^2 + \ldots\ldots + v_n^2} + \ldots\ldots\ldots \frac{v_n^2}{v_1^2 + v_2^2 + \ldots\ldots\ldots + v_n^2}}$$

$$= \sqrt{\frac{v_1^2 + v_2^2 + \ldots\ldots\ldots\ldots\ldots + v_n^2}{v_1^2 + Vv + \ldots\ldots\ldots\ldots + v_n^2}}$$

$$\| u \| = \sqrt{1} = 1$$

Remark *2.1.5:*

The process of finding the unit vector in the direction of v is called normalizing the vector v.

Example *(2):*

Find the unit vector in the direction of $v = (3,-1,2)$ and verify that this vector has lenght 1.

Solution *:*

The unit vector is $u = \frac{v}{\|v\|}$

$$u = \frac{v}{\|v\|} = \frac{(3,-1,2)}{\sqrt{9+1+4}} = \frac{1}{\sqrt{14}}(3,-1,2) = \left(\frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{2}{\sqrt{14}}\right)$$

$$\|u\| = \sqrt{\frac{9}{14} + \frac{1}{14} + \frac{4}{14}} = \sqrt{\frac{14}{14}} = \sqrt{1} = 1$$

u is a unit vector

Definition *2.1.5:*

The distance between two vectors u and v in R^n is

$$d(u,v) = \|u - v\|$$

properties of $d(u,v)$:

(1) $d(u,v) \geq 0$

(2) $d(u,v) = 0$ iff $u = v$

(3) $d(u,v) = d(v,u)$

Example *(3) :*

Find the distance between $u = (0,2,2)$ and $v = (2,0,1)$ is

$$\begin{aligned} d(u,v) &= \|u - v\| \\ &= \|(0-2\ ,\ 2-0\ ,\ 2-1)\| \\ &= \sqrt{(-2)^2 + (2)^2 + (1)^2} \\ &= \sqrt{4+4+1} = \sqrt{9} = 3 \end{aligned}$$

Definition *2.1.7:*

The dot product of $u = (u_1, u_2, \ldots\ldots, u_n)$ and $v = (v_1, v_2, \ldots\ldots v_n)$ in R^n is the scalar quantity

$$\underline{u \cdot v} = u_1 v_1 + u_1 v_2 + \ldots\ldots + u_n v_n$$

which is a scalar not another vector

Example *(4):*

Find the dot product of $u = (1,2,0,-3)$ and $v = (3,-2,4,2)$

$$\begin{aligned} u \cdot v &= 1(3) + 2(-2) + 0(4) + (-3)(2) \\ &= 3 - 4 + 0 - 6 = -7 \end{aligned}$$

Theorem *2.1.8:*

Properties of dot product

if u, v and w are vectors in R^n and c is a scalar, then the following properties are true

(1) $u \cdot v = v \cdot u$

(2) $u \cdot (v + w) = u \cdot v + u \cdot w$

(3) $c(u \cdot v) = (cu) \cdot v = u \cdot (cv)$

(4) $v \cdot v = \|v\|^2$

(5) $v \cdot v \geq 0$ and $v \cdot v = 0$ iff $v = 0$

Proof :

(Home worke)

if R^n is combined with the standard operations of vector addition, scalar multiplicatin, vector length and the dot product its called the Euctidean n-space.

Example *(5):*

Given two vectors u and v in R^n such that $u \cdot u = 39$, $u \cdot v = -3$, $v \cdot v = 79$ evaluate $(u + 2v) \cdot (3u + v)$

Solution *:*

$$\begin{aligned}(u + 2v) \cdot (3u + v) &= u \cdot (3u + v) + (2v) \cdot (3u + v) \\ &= u \cdot (3u) + u \cdot v + (2v) \cdot (3u) + (2v) \cdot v \\ &= 3(u \cdot u) + u \cdot v + 6(v \cdot u) + 2(v \cdot v) \\ &= 3(u \cdot u) + 7(u \cdot v) + 2(v \cdot v) \\ &= 3(39) + 7(-3) + 2(79) \\ &= 254\end{aligned}$$

Theorem *2.1.9: ((The Cauchy-Schwarz Inequality))*

If u and v are vectors in R^n , then

$$|u \cdot v| \leq \|u\| \, \|v\|$$

where $|u \cdot v|$ denotes the absolute value of $u \cdot v$.

Example *(6):*

Verify Cauchy-Schwarz Inequality for $u = (1, -1, 3)$ and $v = (2, 0, -1)$

$u \cdot v = 2 + 0 - 3 = -1$

$\|u\| = \sqrt{1 + 1 + 9} = \sqrt{11}$

$\|v\| = \sqrt{4 + 0 + 1} = \sqrt{5}$

$|u \cdot v| \leq \|u\| \, \|v\|$

$$\begin{aligned}|-1| = 1 &\leq \sqrt{11}\,\sqrt{5} \\ &\leq \sqrt{55} = 7.4\end{aligned}$$

Definition *2.1.10:*

The angle θ between two nonzero vectors in R^n is given by

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|}, 0 \le \theta \le \pi$$

we don't define the angle between zero vectors and other vector.

Example *(7):*

Find the angle θ between $u = (-4, 0, 2, -2)$ and $v = (2, 0, -1, 1)$

$$\cos\theta = \frac{u \cdot v}{\|u\| \cdot \|v\|} = \frac{-8+0-2-2}{\sqrt{24}\sqrt{6}} = \frac{-12}{\sqrt{144}} = -1$$

$\therefore\ \theta = \pi$ so u, v are opposite direction because $u = -2v$

Not that because $\|u\|$ and $\|v\|$ are always positive u, v and $\cos\theta$ will always have the same sign ,

Moreover since the cosine is positive in the first quadrant and negative in second quadrant ,

the sign of the dot product of two vectors can be used to determine whether the angle between them is acute or obtuse as shown.

Definition *2.1.11:*

Two vectors u and v in R^n orthogonal if

$$u \cdot v = 0$$

Even though the angle between zero vector and another vector is not defined ,
its convenient to extend the definition of orthogonality to include the zero vector.
In other words , we say that the vector 0 is orthogonal to every vector

Example *(8):*

a) the vector $u = (1, 0, 0)$ and $v = (0, 1, 0)$ are orthogonal
since $u \cdot v = 1(0) + 0(1) + 0(0) = 0$

b) the vector $u = (3, 2, -1, 4)$ and $v = (1, -1, 1, 0)$ are orthogonal
since $u \cdot v = 3(1) + 2(-1) + (-1)(1) + 4(0) = 3 - 2 - 1 = 0$

Example *(9):*

Determine all vectors in R^n that are orthogonal to $u = (4, 2)$
let $v = (v_1, v_2)$ be orthogonal to u , then

$u \cdot v = (4, 2) \cdot (v_1, v_2)$
$\quad = 4v_1 + 2v_2 = 0$
$\therefore\ 2v_2 = -4v_1$
$\quad v_2 = -2v_1 = -2t$
let $v_1 = t$
$v = (v_1, v_2) = (t, -2t) = t(1, -2)$, $t \in R$
we can use Cauchy-Schwarz Inequality to prove

Theorem *2.1.12: ((The Triangle Inequality))*

if u and v are vectors in R^n , then

$$\|u+v\| \leq \|u\| + \|v\|$$

Theorem *2.1.13: ((Pythagoream Theorem))*

if u and v are vectors in R^n , then u and v are orthogonal *iff*

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

2.2 : Inner product spaces

Here we extends last concepts one step to general vector spaces we accomphih this by using the notion of an Inner product of two vectors the dot product in R^n is called The Euclidean Inner product

$u \cdot v$ = dot product ((Euclidean Inner product in R^n))

$\langle u, v\rangle$ = general Inner product for vector space V.

Definition *of Inner product 2.2.1:*

let u, v and w be vectors in vector space V, and let c be any scalar.
An Inner product on V is a function that associates a real number $\langle u, v\rangle$ with each pair of vectors u and v and satisfies the following axioms:

1) $\langle u, v\rangle = \langle v, u\rangle$
2) $\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle$
3) $c\langle u, v\rangle = \langle cu, v\rangle$
4) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ iff $v = 0$

A vector space V with Inner product is called An Inner Product Space
The Euclidean Inner product is the most important Inner product on R^n

Example *(10):*

Show that the following function defined an Inner product on R^n

$$\langle u, v\rangle = 3u_1 v_1 + 2u_2 v_2$$

Solution *:*

1) $\langle u, v\rangle = 3u_1 v_1 + 2u_2 v_2$
$= 3v_1 u_1 + 2v_2 u_2$
$= \langle v, u\rangle$

2) $\langle u, v + w\rangle = 3u_1(v_1 + w_1) + 2u_2(v_2 + w_2)$
$= 3u_1 v_1 + 3u_1 w_1 + 2u_2 v_2 + 2u_2 w_2$
$= (3u_1 v_1 + 2u_2 v_2) + (3u_1 w_1 + 2u_2 w_2)$
$= \langle u, v\rangle + \langle u, w\rangle$

3) $\langle ku, v\rangle = 3ku_1 v_1 + 2ku_2 v_2$
$= k(3u_1 v_1 + 2u_2 v_2)$
$= k\langle u, v\rangle$

4) $\langle u, u\rangle = 3u_1 u_1 + 2u_2 u_2$
$= 3u_1^2 + 2u_2^2 \geq 0$

$\langle u, u\rangle = 3u_1^2 + 2u_2^2 = 0 \quad iff \quad u_1 = u_2 = 0$
i.e $u = \langle u_1, u_2\rangle = 0$

Example *(11):*

Let f and g be real valued continuous function in the vector space $c[a, b]$ show that

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx$$

defines an Inner product on $c[a,b]$

Solution *:*

1) $\langle f,g\rangle = \int_a^b f(x)g(x)\,dx = \int_a^b g(x)f(x)\,dx = \langle g,f\rangle$

2) $\langle f,g+h\rangle = \int_a^b f(x)(g(x)+h(x))\,dx = \int_a^b f(x)g(x)\,dx + \int_a^b f(x)h(x)\,dx = \langle f,g\rangle + \langle f,h\rangle$

3) $\langle kf,g\rangle = \int_a^b kf(x)g(x)\,dx = k\int_a^b f(x)g(x)\,dx = k\langle f,g\rangle$

4) snice $[f(x)]^2 \geq 0$ for all x, then

$\langle f,f\rangle = \int_a^b [f(x)]^2\,dx \geq 0$

$\langle f,f\rangle = \int_a^b (f(x))^2\,dx = 0 \quad \text{iff} \quad f(x) = 0$

i.e f is the zero function in $c[a,b]$

Theorem *2.2:2:*

let u, v and w be vectors in an inner product space v ,and let c be any real number

1) $\langle 0,v\rangle = \langle v,0\rangle = 0$

2) $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$

3) $\langle u,cv\rangle = c\,\langle u,v\rangle$

Proof :

1) $\langle 0,v\rangle \;=\; \langle v,0\rangle$ by def

$$\begin{aligned}\langle 0,v\rangle &= \langle 0v,v\rangle \\ &= 0\,\langle v,v\rangle \\ &= 0\end{aligned}$$

Definition *2.2.3:*

If u and v are vectors in an inner pruduct space v

1) The norm (or length) of u is $\|u\| = \sqrt{\langle u,u\rangle}$

2) The distance between u and v is $d(u,v) = \|u-v\|$

3) The angle between two nonzero vectors u and v is given by

$$\cos\theta = \frac{\langle u,v\rangle}{\|u\|\,\|v\|}\;,\; 0 \leq \theta \leq \pi$$

4) u and v are orthogonal if $\langle u,v\rangle = 0$

Remark *:*

If $\|v\| = 1$, then v is called a unit vector,Moreover if v is a nonzero vector in an inner product space v,

then the vector $u = \frac{v}{\|v\|}$ is a unit vector and is called the unit vector in the direction of v.

Example *(12):*

let $p(x) = 1 - 2x^2$ and $q(x) = 4 - 2x + x^2$ be polynomials in P_2

find $\langle p, q\rangle$, $\|q\|$,$d(p, q)$,

which pair are orthogonal according to

$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

Solution *:*

$$\langle p, q\rangle = 1(4) + 0(-2) + (-2)(1)$$
$$= 4 - 2 = 2$$

$$\|q\| = \sqrt{\langle q, q\rangle} = \sqrt{4^2 + (-2)^2 + (1)^2} = \sqrt{16 + 4 + 1} = \sqrt{21}$$

$d(p, q)$

$$p(x) - q(x) = 1 - 2x^2 - 4 + 2x - x^2$$
$$= -3x^2 + 2x - 5$$
$$= -5 + 2x - 3x^2$$

$$\|p - q\| = d(p, q) = \sqrt{(3)^2 + (2)^2 + (-3)^2}$$
$$= \sqrt{9 + 4 + 9}$$
$$= \sqrt{22}$$

$\langle p, q\rangle = 2 \neq 0$ is not orthogonal

if $r(x) = x + 2x^2$

$\langle p, q\rangle = 2 \neq 0$

$\langle p, r\rangle = 1(0) + 0(1) - 2(2) = -4 \neq 0$

$\langle q, r\rangle = 4(0) - 2(1) + (1)(2) = -2 + 2 = 0$

so q, r are orthogonal.

Example *(13):*

If $\langle f, g\rangle = \int_0^1 f(x)g(x)\,dx$

find $\|f\|, d(f, g)$ for $f(x) = x,\ g(x) = x^2$ in $c[0, 1]$

Solution *:*

$$\|f\|^2 = \langle f, f\rangle = \int_0^1 (x)(x)\,dx = \int_0^1 x^2\,dx = \left[\ \frac{x^3}{3}\ \right]_0^1$$
$$= \frac{1}{3} - 0 = \frac{1}{3}$$

$$\|f\| = \frac{1}{\sqrt{3}}$$

$$[d(f, g)]^2 = \langle f - g, f - g\rangle$$
$$= \int_0^1 (x - x^2)^2\,dx$$

$$= \int_0^1 (x^2 - 2x^3 + x^4)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{2x^4}{4} + \frac{x^5}{5} \right]_0^1$$

$$= \frac{1}{3} - \frac{1}{2} + \frac{1}{5} = \frac{1}{30}$$

$$d(f,g) = \frac{1}{\sqrt{30}}$$

Remark :

properties of length and distance for R^n olso hold for general Inner produce spaces, if u, and v are vectors in an inner product space then the following properties is true:

properties of norm

a) $\|u\| \geq 0$
b) $\|u\| = o \Leftrightarrow u = 0$
c) $\|cu\| = c\|u\|$
d) $\|u+v\| \leq \|u\| + \|v\|$
triangle inequality

properties of distance

a) $d(u,v) \geq 0$
b) $d(u,v) = 0 \Leftrightarrow u = v$
c) $d(u,v) = d(v,u)$
d) $d(u,v) \leq d(u,w) + d(w,v)$
triangle inequality

Theorem *2.2.4:*

let u, v be vectors in an inner product space v.
1) Cauchy schwarz inequality : $|\langle u,v\rangle| \leq \|u\|\,\|v\|$.
2) Triangle inequality: $\|u+v\| \leq \|u\| + \|v\|$.
3) Pythagoream theorem: u and v are orthogonal iff

$$\|u+v\|^2 = \|u\|^2 + \|v\|^2$$

before we prove this theorem we need to prove the following lemma:

Lemma *2.2.5:*

If a,b,c are real numbers such that $a > 0$
and $a\lambda^2 + 2b\lambda + c \geq 0 \quad \forall\ \lambda \in R$,then

$$b^2 \leq ac$$

Proof :

completing the squeres

$$a\lambda^2 + 2b\lambda + c = a(\lambda^2 + \tfrac{2b}{a}\lambda) + c$$
$$= a(\lambda^2 + \tfrac{2b}{a}\lambda + \tfrac{b^2}{a^2}) + (c - \tfrac{b^2}{a})$$
$$= a(\lambda + \tfrac{b}{a})^2 + (c - \tfrac{b^2}{a})$$
$$= \tfrac{1}{a}(a\lambda + b)^2 + (c - \tfrac{b^2}{a}) \geq 0 \quad \forall\ \lambda$$

this must be true for

$\lambda = \frac{-b}{a}$ thus $c - \frac{b^2}{a} \geq 0$

$$-\frac{b^2}{a} \geq -c$$

$\frac{b^2}{a} \leq c$ and since $a > 0$

$$b^2 \leq ac$$

Proof of Th:2.2.4:

1) If $u = 0$,then $\langle u, v\rangle = \langle 0, v\rangle = 0$

Assume now that $u \neq 0$

then for any scalar t we have

$$0 \leq \|tu + v\|^2 = \langle tu + v, tu + v\rangle$$
$$= t^2\langle u, u\rangle + 2t\langle u, v\rangle + \langle v, v\rangle$$

let $a = \langle u, u\rangle$, $b = \langle u, v\rangle$, $c = \langle v, v\rangle$

$$= at^2 + 2bt + c$$

so by lemma 2.2.5

$$b^2 \leq ac$$
$$\langle u, v\rangle^2 \leq \langle u, u\rangle \langle v, v\rangle$$
$$|\langle u, v\rangle|^2 \leq \|u\|^2 \|v\|^2$$
$$|\langle u, v\rangle| \leq \|u\| \|v\|$$

2) $\|u + v\|^2 = \langle u + v, u + v\rangle$

$$= \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle$$
$$\leq \langle u, u\rangle + 2|\langle u, v\rangle| + \langle v, v\rangle$$
$$\leq \|u\|^2 + 2\|u\| \|v\| + \|v\|^2$$
$$\leq (\|u\| + \|v\|)^2$$

$$\|u + v\| \leq \|u\| + \|v\|$$

3) we note that $\langle u, v\rangle = 0 = \langle v, u\rangle$

$$\|u + v\|^2 = \langle u + v, u + v\rangle$$
$$= \langle u, u\rangle + 2\langle u, v\rangle + \langle v, v\rangle$$
$$= \langle u, u\rangle + 0 + \langle v, v\rangle$$
$$= \|u\|^2 + \|v\|^2$$

Example *(14) :*

Let $f(x) = 1$ and $g(x) = x$ be functions in the vector space $c[0, 1]$with the inner product $\int_0^1 f(x). g(x) dx = \langle f, g\rangle$

Verify Cauchy Schwariz inequality and find $d(f, g)$

We want to prove $|\langle f, g\rangle| \leq \|f\|. \|g\|$

$$\langle f, g\rangle = {}_0\int^1 f(x). g(x) dx = {}_0\int^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\langle f,f\rangle = \|f\|^2 = \int_0^1 dx = [x]_0^1 = 1 \Rightarrow \|f\| = 1$$
$$\langle g,g\rangle = \|g\|^2 = \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3} \Rightarrow \|g\| = 1$$
$$\|f\|\ \ \|g\| = \frac{1}{\sqrt{3}} = 0.577$$
$$|\langle f,g\rangle| \le \|f\|\ \ \|g\| \Rightarrow\ 0.5\ \le 0.577$$
$$\left[d(f,g)\right]^2 = \langle f-g, f-g\rangle = \|f-g\|^2$$
$$= \int_0^1 [f(x) - g(x)]^2\,dx$$
$$= \int_0^1 [1 - x]^2\,dx$$
$$= \int_0^1 [1 - 2x + x^2]^2\,dx$$
$$= \left[x - \frac{2x^2}{2} + \frac{x^3}{3}\right]_0^1 = 1 - 1 + \frac{1}{3} = \frac{1}{3}$$
$$d(f,g) = \frac{1}{\sqrt{3}}$$

2.3 Orthogonal projection in inner product space:

Let u and v be vectors in the plane . If v is nonzero , then we can orthogonally project u and v.

This projection is denoted by $proj_v u$, since $proj_v u$ is a scolar multiple of v, we can write

$$proj_v u = av$$

If $a > 0$ then $\cos\theta > 0$ in (a) , the length of the $proj_v u$ is

$$\|av\| = \|u\|\cos\theta = \frac{\|u\|\ \|v\|\ \cos\theta}{\|v\|} = \frac{\langle u, v\rangle}{\|v\|}$$

Which implies that

$$a = \frac{\langle u, v\rangle}{\|v\|^2} = \frac{\langle u, v\rangle}{\langle v, v\rangle}$$

If $a < 0$ could be shown by the same formula

Definition *:*

Let u , v be vectors in an inner product space v such that $v \neq 0$, then the orthognal projection of u onto v is given by

$$proj_v u = av = \frac{\langle u, v\rangle}{\langle v, v\rangle} v$$

Remark *:*

If v is a unit vector , then $< v, v > = \|v\|^2 = 1$
then $proj_v u = < u, v > v$

Theorem *2.3.1:*

Let u and v be two vectors in an inner product space v, such that $v \neq 0$, then

$$d(u, proj_v u) < d(u, cv), \qquad c \neq \frac{\langle u, v\rangle}{\langle v, v\rangle}$$

Proof :

Let $b = \frac{\langle u, v\rangle}{\langle v, v\rangle}$, then we can write

$$\|u - cv\|^2 = \|(u - bv) + (b - c)v\|^2$$

where $(u - bv), (b - c)v$ are orthogonal
$\langle u - bv, (b - c)v\rangle = b\langle u, v\rangle + bc\langle v, v\rangle - c\langle u, v\rangle - b^2\langle v, v\rangle$
$= (b - c)\langle u, v\rangle + b(c - b)\langle v, v\rangle$
$= (b - c)\frac{\langle u, v\rangle}{\langle v, v\rangle} + b(c - b)\frac{\langle u, v\rangle}{\langle v, v\rangle}\langle v, v\rangle$
$= (b - c)b + b(c - b) = 0$
So $\langle u - bv, (b - c)v\rangle = 0$

by Pythagoream Theorem :
$\| u - bv + (b - c)v\|^2 = \|u - bv\|^2 + \|(b - c)v\|^2$
$= \|u - bv\|^2 + (b - c)^2 \|v\|^2$

Since $b \neq c$ and $v \neq 0$ we know that $(b - c)^2 \|v\|^2 > 0$, Therefore
$\|u - bv\|^2 < \|u - cv\|^2$
$\Rightarrow d(u, bv) < d(u, cv)$

$\Rightarrow d(u, \frac{\langle u,v\rangle}{\langle v,v\rangle} v) < d(u, cv)$
$\Rightarrow d(u, proj_v u) < d(u, cv)$

Example *(15) :*

In R^3, write the Euclidean inner product, find the orthogonal projection of u onto v, where

$u = (3, 1, 2)$ and $v = (7, 1, -2)$

Solution *:*

$proj_v u = \frac{\langle u,v\rangle}{\langle v,v\rangle} v$
$\langle u, v\rangle = 21 + 1 - 4 = 18$
$\langle v, v\rangle = \|v\|^2 = 49 + 1 + 4 = 54$
$proj_v u = \frac{18}{54}(7, 1, -2)$
$= \frac{1}{3}(7, 1, -2)$
$= (\frac{7}{3}, \frac{1}{3}, \frac{-2}{3})$

2.3.2 The Orthogonal Complements:

If v is a plane through the origin of R^3 with the Euclideam inner product, then the set of all vectors that are orthogonal to every vector in v forms the line L through the origin that is porpendicular to v.

In the language of linear algebra we say that the line and the plane are Orthogonal Complement if one another

Definition *2.3.3:*

Let W be a subspace of an inner product space V. A vector u in V is said to be orthogonal to W if it is orthogonal to every vector in W, and the set of all vectors in V that are orthogonal to W is called the orthogonal complement of W

The orthogonal complement of a subspace W is denoted by $W^{\perp}$ read (W perp).

Theorem *2.3.4:*

If W is a subspace of an inner product space V. Then,
(a) $W^{\perp}$ is a subspace of V.
(b) The Only vector common to both W and $W^{\perp}$ is 0.
(c) The orthogonal complement of $W^{\perp}$ is W; that is $(W^{\perp})^{\perp} = W$

Proof :

(a) Note that $\langle 0, w\rangle = 0$ for every $w \in W$, So $W^{\perp}$ contains at least the zero vector.
We want to show that $W^{\perp}$ is closed under addition and scalor multiplication
i.e. we want to show that the sum ot two vectors in $W^{\perp}$ is orthogonal to every vector in W
and similarly for matrix multiplication
Let $u, v \in W^{\perp}$, k any scalor and let $w \in W$
so $\langle u, w\rangle = 0$, $\langle v, w\rangle = 0$ so
$\langle u + v, w\rangle = \langle u, w\rangle + \langle v, w\rangle = 0 + 0 = 0$
So $u + v \in W^{\perp}$
$\langle ku, w\rangle = k\langle u, w\rangle = k0 = 0$
So $ku \in W^{\perp}$
$\therefore$ $W^{\perp}$ is a subspace of V.

Remark *:*

W and $W^{\perp}$ are orthogonal Complement

Theorem *2.3.5:*

If A is an $m \times n$ matrix then,
(a) The null space of A and the row space of A are orthogonal complements in R^n with respect to the standard Euclidean inner product.
(b) The null space of A^t and the column space of A are orthogonal complements in R^m with respect to the standard Euclidean inner product.

Example *(16):*

Let W be the subspace of R^5 spanned by the vectors
$w_1 = (2, 2, -1, 0, 1)$; $w_2 = (-1, -1, 2, -3, 1)$; $w_3 = (1, 1, -2, 0, -1)$ and $w_4 = (0, 0, 1, 1, 1)$
Find the basis for the orthogonal complement of W.

Solution *:*

The space W spanned by w_1, w_2, w_3 and w_4 to the same as the row spase of the matrix

$$A = \begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

The null space of A is the orthogonal complement of W

$$\begin{pmatrix} -1 & -1 & 2 & -3 & 1 \\ 2 & 2 & -1 & 0 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_1 + x_2 + x_5 = 0 \Rightarrow x_1 = -x_2 - x_5$ Let $x_2 = t,\ x_5 = s$
$x_3 + x_5 = 0 \Rightarrow x_3 = -x_5 = -s$
$x_4 = 0 \Rightarrow x_4 = 0$ $so\ \ x_1 = -t - s$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} -t-s \\ t \\ -s \\ 0 \\ s \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = u_1 + u_2$$

u_1, u_2 form the basis for all nullspace of W.
i.e. u_1, u_2 form basis for the orthogonal cpmplement of W The basis of the row space is

$$\begin{pmatrix} 2 & 2 & -1 & 0 & 1 \\ -1 & -1 & 2 & -3 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & -1 \\ 2 & 2 & -1 & 0 & 1 \\ 1 & 1 & -2 & 0 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 3 & -6 & 3 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & -2 & 3 & -1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$v_1 = (1, 1, -2, 3, -1)$
$v_2 = (0, 0, 1, -2, 1)$
$v_3 = (0, 0, 0, 1, 0)$
basis for row space of W
$u_1 . v_1 = 0, \quad u_1 . v_2 = 0, \quad u_1 . v_3 = 0$
$u_2 . v_1 = 0, \quad u_2 . v_2 = 0, \quad u_2 . v_3 = 0$

Problem Set II

(1) Find (a) $\langle u, v\rangle$, (b) $\|u\|$, and (c) $d(u, v)$ for the given inner product defined in R^n

(i) $u = (3,4), v = (5,-12), \langle u, v\rangle = u \cdot v$

(ii) $u = (-4,3),\ v = (0,5), \langle u, v\rangle = 3u_1 v_1 + u_2 v_2$

(iii) $u = (0,9,4),\ v = (9,-2,-4), \langle u, v\rangle = u \cdot v$

(iv) $u = (8,0,-8),\ v = (8,3,16),\ \langle u, v\rangle = 2u_1 v_1 + 3u_2 v_2 + u_3 v_3$

(2) Use the given functions f and g in $C[-1,1]$ to find (a) $\langle f, g\rangle$, (b) $\|f\|$, and (c) $d(f, g)$ for the inner product given by

$$\langle f, g\rangle = \int_0^1 f(x).\, g(x)\, dx$$

(i) $f(x) = x^2, \quad g(x) = x^2 + 1$

(ii) $f(x) = x, \quad g(x) = e^x$

(3) Use the inner product

$$\langle A, B\rangle = 2a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

To find (a) $\langle A, B\rangle$, (B) $\|A\|$, and (c) $d(A, B)$ for the given matrices in $M_{2,2}$

$$A = \begin{bmatrix} -1 & 3 \\ 4 & -2 \end{bmatrix},\quad B = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$$

(4) Use the inner product

$$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$$

To find (a) $\langle p, q\rangle$, (B) $\|p\|$, and (c) $d(p, q)$ for the given polynomials in P_2

$$p(x) = 1 - x + 3x^2, \quad q(x) = x - x^2$$

(5) State why $\langle u, v\rangle$ is not an inner product for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in R^2

(i) $\langle u, v\rangle = u_1 v_1$

(ii) $\langle u, v\rangle = u_1 v_1 - u_2 v_2$

(6) Find the angle between the given vectors

(i) $u = (3,4), \quad v = (5,-12),\ \langle u, v\rangle = uv$

(ii) $u = (1,1,1), \quad v = (2,-2,2),\ \langle u, v\rangle = u_1 v_1 + 2u_2 v_2 + u_3 v_3$

(iii) $f(x) = x, \quad g(x) = x^2,\ \langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\, dx$

(7) Verify (a) the Cauchy - Schwarz Inequality and (b) the Triangle Inequality

(i) $u = (5, 12), \quad v = (3, 4), \ \langle u, v\rangle = u \cdot v$

(ii) $p(x) = 2x, \quad q(x) = 3x^2 + 1, \ \langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$

(iii) $f(x) = \sin x, \quad g(x) = \cos x, \ \langle f, g\rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx$

(8) Show that f and g are orthogonal in the inner product space $C[a, b]$ with the inner product given by

$$\langle f, g\rangle = \int_a^a f(x) g(x)\, dx$$

$$C[-1, 1], \ f(x) = x, \quad g(x) = \tfrac{1}{2}(5x^3 - 3x)$$

(9) (a) find $proj_v u$, (b) find $proj_u v$

$u = (1, 2), \quad v = (2, 1)$

(10) Find (a) $proj_v u$, and (b) $proj_u v$

$u = (1, 3, -2), \quad v = (0, -1, 1)$

(11) Find the orthogonal projection of f onto g. Use the inner product in $C[a, b]$ given by

$$\langle f, g\rangle = \int_a^a f(x) g(x)\, dx$$

$C[0, 1], \ f(x) = x, \quad g(x) = e^x$

(12) Prove that $\|u + v\|^2 + \|u - v\|^2 = 2\|u\|^2 + 2\|v\|^2$ for any vectors u and v in an inner product space V.

(13) Let u and v be nonzero vectors in an inner product space V. Prove that $proj_v u$ is orthogonal to V.

(14) Let $A = \begin{pmatrix} 1 & 2 & -1 & 2 \\ 3 & 5 & 0 & 4 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ (a) Find bases for the row space and null spase of A

(b) Verify that every vector in row space is orthogonal to every vector in the null space

(15) Find a basis for the orthogonal complement of the subspace of R^n spanned by the vectors

(a) $v_1 = (1, -1, 3), v_2 = (5, -4, -4), v_3 = (7, -6, 2)$

(b) $v_1 = (1, 4, 5, 6, 9)$, $v_2 = (3, -2, 1, 4, -1)$, $v_3 = (-1, 0, -1, -2, -1)$, $v_4 = (2, 3, 5, 7, 5)$

(16) Let V be an inner product space, show that if u and v are orthogonal vectors in V such that

$$\|u\| = \|v\| = 1 \text{ then } \|u - v\| = \sqrt{2}$$

2.4 : Orthonormal Basis, Gram - Schmidt Process; QR - Decomposition

Definition *2.4.1:*

A set of vectors in an inner product space is called an orthogonal set if all pairs of distinct vectors in the set are orthogonal. An orthogonal set in which each vectors has norm 1 is called an orthonormal .

Remark *:*

For $S = \{v_1, v_2, \ldots\ldots, v_n\}$ this definition has the following for

orthogonal : orthonormal

$$(1)\ \langle v_i, v_j\rangle = 0,\ i \neq j \qquad \vdots \qquad \left\{ \begin{array}{c} (1)\ \langle v_i, v_j\rangle = 0, i \neq j \\ (2)\ \|v_i\| = 1,\ \ i = 1,2,3,\ldots\ldots, n \end{array} \right\}$$

if S is a basis then its called an orthogonal basis or an orthonormal basis

Example *(17) :*

Show that the following set is an orthonormal basis for R^3

$$S = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right), \left(\frac{-\sqrt{2}}{6}, \frac{\sqrt{2}}{6}, \frac{2\sqrt{2}}{3}\right), \left(\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}\right)\right\}$$

Solution *:*

First we show that the three vectors are mutually orthogonal

$\langle v_1, v_2\rangle = \frac{-1}{6} + \frac{1}{6} + 0 = 0$

$\langle v_1, v_3\rangle = \frac{2}{3\sqrt{2}} - \frac{2}{3\sqrt{2}} + 0 = 0$

$\langle v_2, v_3\rangle = \frac{-\sqrt{2}}{9} - \frac{\sqrt{2}}{9} + \frac{2\sqrt{2}}{9} = 0$

$\|v_1\| = \sqrt{\frac{1}{2} + \frac{1}{2} + 0} = 1$

$\|v_2\| = \sqrt{\frac{2}{36} + \frac{2}{36} + \frac{8}{9}} = \sqrt{\frac{2+2+32}{36}} = 1$

$\|v_3\| = \sqrt{\frac{4}{9} + \frac{4}{9} + \frac{1}{9}} = 1$

So S is an orthonormal set of vectors

Theorem *2.4.2 :*

If $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is an orthogonal set of non-zero vectors in an inner product space v,
then S in linearly independent

Proof :

We need to shwo that the vectors equation

$$c_1 v_1 + c_2 v_2 + \ldots\ldots + c_n v_n = 0$$

implies that $c_1 = c_2 = \ldots\ldots = c_n = 0$. To do this, we take the inner product of the left side of the equation with each vector in S. That is, for each i

$$\langle (c_1 v_1 + c_2 v_2 + \ldots\ldots + c_i v_i + \ldots\ldots\ldots + c_n v_n), v_i \rangle = \langle 0, v_i \rangle = 0$$
$$c_1 \langle v_1, v_i \rangle + c_2 \langle v_2, v_i \rangle + \ldots\ldots + c_i \langle v_i, v_i \rangle + \ldots\ldots + c_n \langle v_n, v_i \rangle = 0$$

Now since S is orthogonal , $\langle v_i, v_i \rangle = 0$ for $i \neq j$, and thus the equation reduces to $c_i \langle v_i, v_i \rangle = 0$

But because each vector in S is non-zero, we know that $\langle v_i, v_i \rangle = \|v_i\|^2 \neq 0$

Hence every c_i must be zero and the set must be linearly independent

Corollary *2.4.3 :*

If V is an inner product space of dimension n, then any orthogonal set of n vectors is a basis of V.

Example *(18) :*

Show that the following set is a basis for R^4

$$S = \{(2,3,2,-2),(1,0,0,1),(-1,0,2,1),(-1,2,-1,1)\}$$

Solution *:*

$\langle v_1, v_2 \rangle = 2 + 0 + 0 - 2 = 0$

$\langle v_1, v_3 \rangle = -2 + 0 + 4 - 2 = 0$

$\langle v_1, v_4 \rangle = -2 + 6 - 2 - 2 = 0$

$\langle v_2, v_3 \rangle = -1 + 0 + 0 + 1 = 0$

$\langle v_2, v_4 \rangle = -1 + 0 + 0 + 1 = 0$

$\langle v_3, v_4 \rangle = 1 + 0 - 2 + 1 = 0$

Thus S is orthogonal , and by the corollary above its a basis for R^4

Coordinates relative to orthonormal basis :

Theorem *2.4.4 :*

If $S = \{v_1, v_2, \ldots\ldots, v_n\}$ that is an orthogonal basis for an inner product space V and u is any vector in V then,

$$u = \langle u, v_1 \rangle v_1 + \langle u, v_2 \rangle v_2 + \ldots\ldots + \langle u, v_n \rangle v_n \qquad (*)$$

where (*) is the coordinate representation of u with respect to S . i.e.

$$[u]_S = (u)_S = \{\langle u, v_1 \rangle, \langle u, v_2 \rangle, \ldots\ldots, \langle u, v_n \rangle\}$$

Proof :

Since $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is a basis , a vector u can be expressed in the form

$$u = k_1 v_1 + k_2 v_2 + \ldots\ldots + k_n v_n$$

we will show that $k_i = \langle u, v_i \rangle$ for $i = 1, 2, \ldots\ldots, n$
For each vector v_i in S we have

$$\langle u, v_i \rangle = \langle k_1 v_1 + k_2 v_2 + \ldots\ldots + k_n v_n, v_i \rangle \qquad (**)$$
$$= k_1 \langle v_1, v_i \rangle + k_2 \langle v_2, v_i \rangle + \ldots\ldots + k_n \langle v_n, v_i \rangle$$

Since $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is orthonormal set, we have $\langle v_i, v_i \rangle = \|v_i\|^2 = 1$ and $\langle v_j, v_i \rangle = 0$ for $i \neq j$
Therefore (**) for $\langle u, v_i \rangle$ simiphlies to $\langle u, v_i \rangle = k_i$

So $(u)_S = \{\langle u, v_1 \rangle, \langle u, v_2 \rangle, \ldots\ldots, \langle u, v_n \rangle\}$
Which are called : Fourier Coefficients of u relative to S

Example *(19) :*

Let $v_1 = (0, 1, 0), v_2 = (\frac{-4}{5}, 0, \frac{3}{5}), v_3 = (\frac{3}{5}, 0, \frac{4}{5})$ be an orthonormal basis for R^3.
Express the vector $u = (1, 1, 1)$ as a linear combination of the vectors in S , $S = \{v_1, v_2, v_3\}$, and find the coordinate vector $(u)_S$

Solution *:*

$\langle u, v_1 \rangle = 1, \ \langle u, v_2 \rangle = \frac{-1}{5}, \quad \langle u, v_3 \rangle = \frac{7}{5}$

$u = v_1 - \frac{1}{5} v_2 + \frac{7}{5} v_3$

$(1, 1, 1) = (0, 1, 0) - (\frac{-4}{5}, 0, \frac{3}{5}) + (\frac{3}{5}, 0, \frac{4}{5})$
$(u)_S = (1, \frac{-1}{5}, \frac{7}{5})$

Theorem *2.4.5 :*

If S is an orthonormal basis for n - dimenional inner product space, and if

$$(u)_s = (u_1, u_2, \ldots\ldots, u_n) \quad \text{and} \quad (v)_s = (v_1, v_2, \ldots\ldots\ldots, v_n)$$

then

(a) $\|u\| = \sqrt{(u_1)^2 + (u_2)^2 + \ldots\ldots + (u_n)^2}$

(b) $d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + \ldots\ldots + (u_n - v_n)^2}$

(c) $\langle u, v\rangle = u_1 v_1 + u_2 v_2 + \ldots\ldots + u_n v_n$

Example *(20):*

If R^3 has the Euclidean inner product, then the norm of $u = (1, 1, 1)$ is $\|u\| = \sqrt{1 + 1 + 1} = \sqrt{3}$

from last example $(u)_S = (1, \frac{-1}{5}, \frac{7}{5})$

we can calculate $\|u\| = \sqrt{1 + \frac{1}{25} + \frac{49}{25}} = \sqrt{\frac{25+1+49}{25}} = \sqrt{\frac{75}{25}} = \sqrt{3}$

2.4.5. Coordinates Relative to Orthoginal Basia:

If $S = \{v_1, v_2, \ldots\ldots, v_n\}$ is an orthogonal basis for a vector space v, then normalizing each of these vectors yields the orthonormal basis

$$S' = \left\{ \frac{v_1}{\|v_1\|}, \frac{v_2}{\|v_2\|}, \ldots\ldots, \frac{v_n}{\|v_n\|} \right\}$$

Thus u is any vector in v it follows that

$$u = \frac{\langle u, v_1 \rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2 \rangle}{\|v_2\|^2} v_2 + \ldots\ldots + \frac{\langle u, v_n \rangle}{\|v_n\|^2} v_n$$

Orthogonal projection :

We shall now develop some results that will help to construct orthogonal and orthonormal bases for inner product space. In R^2 and R^3 with the Euclidean inner product its evident that if W is a line or a plane through the origin then each vector u in the space can be expressed as a sum

$$u = w_1 + w_2$$

where w_1 is in W and w_2 is perpendicular to W

Theorem *2.4.6: Projection Theorem:*

If W is a finite dimensional subspace of an inner product space V, then every vector u in V can be expressed in exactly are way as

$$u = w_1 + w_2$$

where w_1 is in W and w_2 is perpendicular to $W^{\perp}$

The vector w_1 in the projection theorem is called the orthogonal projection of u on W and is denoted by $proj_w u$.

The vector w_2 is called the component of u orthogonal to W and is denoted by $proj_{W^{\perp}}\, u$. Thus

$$u = w_1 + w_2$$
$$= proj_W\, u. + proj_{W^{\perp}}\, u.$$

Since $w_2 = u - w_1$ it follows that $proj_{W^{\perp}}\, u. = u - proj_W\, u$. So

$$u = proj_W\, u. + (u - proj_W\, u).$$

Theorem *2.4.7 :*

Let W be a finite dimensional subspace of an inner product space V.

(a) If $\{v_1, v_2, \ldots\ldots, v_r\}$ is an orthonormal basis for W, and u is any vector in V, then

$$proj_W\, u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2 + \ldots\ldots\ldots + \langle u, v_r\rangle v_r$$

(b) If $\{v_1, v_2, \ldots\ldots, v_r\}$ is an orthogonall basis for W, and u is any vector in V, then

$$proj_W\, u = \frac{\langle u, v_1\rangle}{\|v_1\|^2} v_1 + \frac{\langle u, v_2\rangle}{\|v_2\|^2} v_2 + \ldots\ldots\ldots + \frac{\langle u, v_r\rangle}{\|v_r\|^2} v_r$$

Example *(21):*

Let R^3 with Euclidean inner product and let W be the subspace spanned by the orthonormal vectors

$v_1 = (0, 1, 0),\ v_2 = (\frac{-4}{5}, 0, \frac{3}{5})$, Find $proj_W\, u$ of $u = (1, 1, 1)$ on W

Solution *:*

$proj_W\, u = \langle u, v_1\rangle v_1 + \langle u, v_2\rangle v_2$

$= 1(0, 1, 0) + (-\frac{1}{5})(\frac{-4}{5}, 0, \frac{3}{5})$

$= (\frac{4}{25}, 1, \frac{-3}{25})$

The component of u orthogonal to W is

$proj_{W^{\perp}}\, u. = u - proj_W\, u$

$= (1, 1, 1) - (\frac{4}{25}, 1, \frac{-3}{25})$

$= (\frac{21}{25}, 0, \frac{28}{25})$

2.4.8: Gram - Schmidt orthonormalization process

Theorem *2.4.9:*

(1) Let $B = \{v_1, v_2, \ldots\ldots, v_n\}$ be a basis for an inner product space V.
(2) Let $B' = \{w_1, w_2, \ldots\ldots, w_n\}$, where w_i is given by

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \\
w_3 &= v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 \\
&\vdots \\
w_n &= v_n - \frac{\langle v_n, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_n, w_2\rangle}{\langle w_2, w_2\rangle} w_2 - \ldots\ldots\ldots - \frac{\langle v_n, w_{n-1}\rangle}{\langle w_{n-1}, w_{n-1}\rangle} w_{n-1}
\end{aligned}
$$

Then B' is an orthogonal basis for V.
(3) Let $u_i = \dfrac{w_i}{\| w_i \|}$. Then the set $B'' = \{u_1, u_2, \ldots\ldots, u_n\}$ is an orthonormal basis for V.

Example *(22):*

Apply the Gram - Shmidt orthormalization process to the following basis of R^3
$B = \{(1,1,0), (1,2,0), (0,1,2)\}$

Solution *:*

$w_1 = v_1 = (1,1,0)$

$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$

$= (1,2,0) - \frac{3}{2}(1,1,0) = (-\frac{1}{2}, \frac{1}{2}, 0)$

$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2$

$= (0,1,2) - \frac{1}{2}(1,1,0) - \frac{\frac{1}{2}}{\frac{1}{2}}(-\frac{1}{2}, \frac{1}{2}, 0) = (0,0,2)$

The set $B' = \{w_1, w_2, w_3\}$ is an orthogonal basis for R^3, Normalizing each vector in B' produces

$u_1 = \dfrac{w_1}{\| w_1 \|} = \frac{1}{\sqrt{2}}(1,1,0) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right)$

$$u_2 = \frac{w_2}{\| w_2 \|} = \frac{1}{\frac{1}{\sqrt{2}}}(-\tfrac{1}{2}, \tfrac{1}{2}, 0) = \left(-\tfrac{\sqrt{2}}{2}, \tfrac{\sqrt{2}}{2}, 0\right)$$
$$u_3 = \frac{w_3}{\| w_3 \|} = \tfrac{1}{2}(0,0,2) = (0,0,1)$$

Thus $B'' = \{u_1, u_2, \ldots\ldots, u_n\}$ is an orthonormal basis for R^3,

This is an alternative form of the Gram - Schmidt orthonormalization proces has the following steps:

$$u_1 = \frac{w_1}{\| w_1 \|} = \frac{v_1}{\| v_1 \|}$$
$$u_2 = \frac{w_2}{\| w_2 \|} \text{ where } w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$
$$u_3 = \frac{w_3}{\| w_3 \|} \text{ where } w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$$
$$\vdots$$
$$u_n = \frac{w_n}{\| w_n \|} \text{ where } w_n = v_n - \langle v_n, u_1 \rangle u_1 - \ldots\ldots\ldots - \langle v_n, u_{n-1} \rangle u_{n-1}$$

Example *(23):*

Apply G.S.O.P to the basis $\{1, x, x^2\}$ in P_2 using the inner product:

$$\langle p, q \rangle = \int_{-1}^{1} p(x) q(x)\, dx$$

Solution *:*

Let $B = \{1, x, x^2\} = \{v_1, v_2, v_3\}$, then

$$w_1 = v_1 = 1$$

$$w_2 = v_2 - \tfrac{\langle v_2, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= x - \frac{\int_{-1}^{1} x dx}{\int_{-1}^{1} dx} = x - \left[\tfrac{x^2}{2x} \right]_{-1}^{1} = x(\tfrac{1}{2} + \tfrac{1}{2}) = x$$

$$w_3 = v_3 - \tfrac{\langle v_3, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \tfrac{\langle v_3, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2$$

$$= x^2 - \frac{\int_{-1}^{1} x^2\,dx}{\int_{-1}^{1} dx} - \frac{\int_{-1}^{1} x^3\,dx}{\int_{-1}^{1} x^2\,dx}\,x$$

$$= x^2 - \frac{\left[\frac{x^3}{3}\right]_{-1}^{1}}{[x]_{-1}^{1}} - \frac{\left[\frac{x^4}{4}\right]_{-1}^{1}}{\left[\frac{x^3}{3}\right]_{-1}^{1}}\,x = x^2 - \frac{1}{3}$$

$$B' = \{w_1, w_2, w_3\}$$

$$u_1 = \frac{w_1}{\| w_1 \|} = \frac{1}{\sqrt{\langle w_1, w_1 \rangle}} = \frac{1}{\sqrt{\int_{-1}^{1} dx}} = \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{w_2}{\| w_2 \|} = \frac{x}{\sqrt{\int_{-1}^{1} x^2\,dx}} = \sqrt{\frac{3}{2}}\,x$$

$$u_3 = \frac{w_3}{\| w_3 \|} = \frac{x^2 - \frac{1}{3}}{\sqrt{\int_{-1}^{1} \left(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\right) dx}} = \frac{3\sqrt{5}}{2\sqrt{2}}(3x^2 - 1)$$

Example *(24):*

Consider the following basis of Euclidean space R^3

$\{v_1 = (1,1,1), v_2 = (0,1,1), v_3 = (0,0,1)\}$

We use the Gram-Schmidt orthogonalization process to transform $\{v_i\}$ into an orthonormal basis $\{u_i\}$.

First we normalize v_1 i.e. we set

$$u_1 = \frac{v_1}{\| v_1 \|} = \frac{(1,1,1)}{\sqrt{3}} = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

Next we set

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1 = (0,1,1) - \frac{2}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \left(\frac{-2}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

and then we normalize w_2, i.e. we get

$$u_2 = \frac{w_2}{\| w_2 \|} = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$$

Finally we set

$$w_3 = v_3 - \langle v_3, u_1 \rangle u_1 - \langle v_3, u_2 \rangle u_2$$

$$= (0,0,1) - \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) - \frac{1}{\sqrt{6}}\left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) = \left(0, \frac{-1}{2}, \frac{1}{2}\right)$$

and then we normalize w_3 :

$$u_3 = \frac{w_3}{\| w_3 \|} = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

The required orthonormal basis of R^3 is

$$\left\{u_1 = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), u_2 = \left(\frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right), u_3 = \left(0, \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}$$

Example *(25):*

Find an orthonormal basis for the solution space of the following homogeneous system of linear equations

$$x_1 + x_2 + \quad +7x_4 = 0$$
$$2x_1 + x_2 + 3x_3 + 6x_4 = 0$$

Solution *:*

The ougmented matrix for this system reduces as follows

$$\begin{bmatrix} 1 & 1 & 1 & 7 & 0 \\ 2 & 1 & 2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 & 0 \\ 0 & 1 & -2 & 8 & 0 \end{bmatrix}$$

Let $x_3 = s$ and $x_4 = t$ then

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2s + t \\ 2s - 8t \\ s \\ t \end{pmatrix}$$

$$= s\begin{pmatrix} -2 \\ 2 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -8 \\ 0 \\ 1 \end{pmatrix}$$

Therfore one basis for the solution space is:

$$B = \{v_1, v_2\} = \{(-2, 2, 1, 0), (1, -8, 0, 1)\}$$

To find the othonormal basis $B' = \{u_1, u_2\}$, we use the alternative form of the G.S.O.P. as follows

$$u_1 = \frac{v_1}{\| v_1 \|} = \frac{1}{3}(-2, 2, 1, 0) = \left(-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, 0\right)$$

$$w_2 = v_2 - \langle v_2, u_1 \rangle u_1$$

$$= (1,-8,0,1) - \left[(1,-8,0,1).\left(-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, 0\right) \right]\left(-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, 0\right)$$

$$= (1,-8,0,1) - (4,-4,-2,0) = (-3,-4,2,1)$$

$$u_2 = \frac{w_2}{\| w_2 \|} = \frac{1}{\sqrt{30}}(-3,-4,2,1) = \left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right)$$

$$B' = \left\{ \left(-\tfrac{2}{3}, \tfrac{2}{3}, \tfrac{1}{3}, 0\right), \left(\frac{-3}{\sqrt{30}}, \frac{-4}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{1}{\sqrt{30}} \right) \right\}$$

2.5. QR DeComposition:

If A is an $m \times n$ matrix with linearly independent column vectors, and if Q is the matrix with orthonormal column vectors that results from applying the Gram-Schmidt Process to the column vectors of A, what relationship, if any, exists between A and Q?

To solve, this problem, suppose that the column vectors of A are $u_1, u_2, \ldots\ldots, u_n$ and the orthonormal column vectors of Q are $q_1, q_2, \ldots\ldots, q_n$: thus

$$A = [u_1 | u_2 | \ldots\ldots | u_n] \text{ and } Q = [q_1 | q_2 | \ldots\ldots | q_n]$$

it follows from Theorem 2.4.4. that $u_1, u_2, \ldots\ldots, u_n$ are experssible in terms of $q_1, q_2, \ldots\ldots, q_n$

$$\begin{aligned} u_1 &= \langle u_1, q_1 \rangle q_1 + \langle u_1, q_2 \rangle q_2 + \ldots\ldots\ldots + \langle u_1, q_n \rangle q_n \\ u_2 &= \langle u_2, q_1 \rangle q_1 + \langle u_2, q_2 \rangle q_2 + \ldots\ldots\ldots + \langle u_2, q_n \rangle q_n \\ &\vdots \\ u_n &= \langle u_n, q_1 \rangle q_1 + \langle u_n, q_2 \rangle q_2 + \ldots\ldots\ldots + \langle u_n, q_n \rangle q_n \end{aligned}$$

As we know that the j^{th} column vector of a matrix product is a linear combination of the column vectors of the first factor with coefficients coming from the j^{th} column of the second factor,

it follows that these relationship can be expressed in matrix form as

$$[u_1 | u_2 | \ldots\ldots | u_n] = [q_1 | q_2 | \ldots\ldots | q_n] = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \ldots. & \langle u_n, q_1 \rangle \\ \langle u_1, q_2 \rangle & \langle u_2, q_2 \rangle & \ldots. & \langle u_n, q_2 \rangle \\ \vdots & & & \\ \langle u_1, q_n \rangle & \langle u_2, q_n \rangle & \ldots. & \langle u_n, q_n \rangle \end{bmatrix}$$

Or more briefly as $A = QR$

However its a property of Gram-Schmidt Process that for $j \geq 2$,
the vector q_i is orthogonal to $u_1, u_2, \ldots\ldots, u_{j-1}$, thus all entries below the main diagonal

of R are zero

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \cdots & \langle u_n, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \cdots & \langle u_n, q_2 \rangle \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & \langle u_n, q_n \rangle \end{bmatrix}$$

Theorem *2.5.1:*

If A is an $m \times n$ matrix with linearly independent column vectors, then A can be factored as

$$A = QR$$

Where Q is an $m \times n$ matrix with orthonormal column vectors and R is an $n \times n$ invertible upper triangular matrix.

Remark *:*

If A is an $n \times n$ matrix then the invertibilty of A is equivalent to linear independence of the column vectors. Thus,

every invertible matrix has a QR-decomposition

Example *(26):*

Find the QR-decompostion of

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Solution *:*

The column vectors of A are

$$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, u_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, u_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Applying G.S.O.P. it yields to the following orthonormal vector

$$q_1 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, q_2 = \begin{bmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, q_3 = \begin{bmatrix} 0 \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$R = \begin{bmatrix} \langle u_1, q_1 \rangle & \langle u_2, q_1 \rangle & \langle u_n, q_1 \rangle \\ 0 & \langle u_2, q_2 \rangle & \langle u_n, q_2 \rangle \\ 0 & 0 & \langle u_n, q_n \rangle \end{bmatrix} = \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Thus the QR-DeComposition of A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{-2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{3}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$A = QR$$

2.6: Applications of inner product space

Definition *2.6.1:*

Cross product of two vectors:
Let $u = u_1 i + u_2 j + u_3 k$ and $v = v_1 i + v_2 j + v_3 k$ be vectors in R^3.
The cross product of u and v is the vector

$$u \times v = (u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - u_2 v_1) k$$

Remark *:*

The cross product is defined only for vectors in R^3. We do not define the cross product of two vectors in R^2.
or of vectors in $R^n,\ n > 3$

A convenient way to remember the formula for the cross product $u \times v$ is to use the following determinat form

$$u \times v = \begin{vmatrix} i & j & k \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{matrix} \\ \Rightarrow \text{ Component of u} \\ \Rightarrow \text{ Component of v} \end{matrix}$$

Technically its not a det. because the enteries are not all real numbers.

$$u \times v = (u_2 v_3 - u_3 v_2) i - (u_1 v_3 - u_3 v_1) j + (u_1 v_2 - u_2 v_1) k$$
$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} i - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} j + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} k$$

Example *(27):*

Given $u = i - 2j + k$ and $v = 3i + j - 2k$ find the following
(a) $u \times v$ (b) $v \times u$ (c) $v \times v$

Solution *:*

$$\text{(a) } u \times v = \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix} = (4 - 1) i - (-2 - 3) j + (1 + 6) k = 3i + 5j + 7k$$

$$(b)\ v\times u = \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix} = (1-4)i-(3+2)j+(-6-1)k = -3i-5j-7k$$

$$(c)\ v\times v = \begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = (-2+2)i-(-6+6)j+(3-3)k = 0i+0j+0k = 0$$

So $u\times v = -(v\times u)$ and $v\times v = 0$

Theorem *2.6.2:*

If u, v and w are vectors in R^3. and c is scalar, then the following properties are true:

(1) $u\times v = -(v\times u)$

(2) $u\times(v+w) = (u\times v)+(u\times w)$

(3) $c(u\times v) = cu\times v = u\times cv$

(4) $u\times 0 = 0\times u = 0$

(5) $u\times u = 0$

(6) $u\cdot(v\times w) = (u\times v)\cdot w$

The proof is homework

Theorem *2.6.3:*

If u, v and are nonzero vectors in R^3, then the following properties are true:

(1) $u\times v$ is orthogonal to both u and v.

(2) The angle θ between u and v is given by:

$$\|u\times v\| = \|u\|\,\|v\|\sin\theta$$

(3) u and v are parallel iff $u\times v = 0$

(4) The parallelogram having u and v adjacent sides has an area of $\|u\times v\|$

Proof :

(4) Let $u,\ v$ be the adjacent sides of a parallelogram, by (2) the area of it is given by

$$Area = \underbrace{\|u\|}_{Base}\ \underbrace{\|v\|\sin\theta}_{hight}$$

$$= \|u\times v\|$$

Example *(28):*

Find a unit vector that is orthogonal to both
$u = i - 4j + k, \quad v = 2i + 3j$

Solution *:*

We know that $u \times v$ is orthogonal to u and v

$$u \times v = \begin{vmatrix} i & j & k \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} = -3i + 2j + 11k$$

by dividing by the length of $u \times v$
$\| u \times v \| = \sqrt{(-3)^2 + (2)^2 + (11)^2} = \sqrt{134}$
we obtain the unit vector
$\frac{u \times v}{\| u \times v \|} = \frac{-3}{\sqrt{134}} i + \frac{2}{\sqrt{134}} j + \frac{11}{\sqrt{134}} k$

Which is orthogonal to both u and v

Example *(29):*

Find the area of the parallelogram that has
$u = -3i + 4j + k$ and $v = -2i + 6k$
as adjacent sides

Solution *:*

$\| u \times v \|$ is the area

$$u \times v = \begin{vmatrix} i & j & k \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} = 26i + 18j + 6k$$

$$\| u \times v \| = \sqrt{(26)^2 + (18)^2 + 36} = \sqrt{1036} \approx 32.19 \ unit^2$$

Problem IV

(1) Find the area of the paralldogram that has the given vectors as adjacent sides:
(a) $u = (1,0,0), \quad v = (0,1,0)$
(b) $u = i + j + k, \quad u = 2i + j - k$

(2) Find the area of the paralldogram that has the given vectors as adjacent sides
(a) $\underline{u} = (3,2,-1), \quad \underline{v} = (1,2,3)$
(b) $\underline{u} = (2,-1,0), \quad \underline{v} = (-1,2,0)$

Problem Set III

(1) Determine whether the set of vectors in R^n is orthogonal, orthonormal, or neither

(i) $\{(\frac{3}{5}, \frac{4}{5}), (\frac{-4}{5}, \frac{3}{5})\}$

(ii) $\left\{\left(\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}\right), \left(\frac{-\sqrt{6}}{6}, \frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{6}\right), \left(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{-\sqrt{3}}{3}\right)\right\}$

(iii) $\left\{\left(\frac{\sqrt{2}}{2}, 0, 0, \frac{\sqrt{2}}{2}\right), \left(0, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 0\right), (\frac{-1}{2}, \frac{1}{2}, \frac{-1}{2}, \frac{1}{2})\right\}$

(2) Verify that $\{1, x, ^2, x^3\}$ is an orthonormal basis for P_3 with the inner product

$$\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3$$

(3) Find the coordinates of x relative to the orthonormal basis B in R^n

(i) $B = \left\{\left(\frac{-2\sqrt{13}}{13}, \frac{3\sqrt{13}}{13}\right), \left(\frac{3\sqrt{13}}{13}, \frac{2\sqrt{13}}{13}\right)\right\}, x = (1,2)$

(ii) $B = \{(\frac{3}{5}, \frac{4}{5}, 0), (\frac{-4}{5}, \frac{3}{5}, 0), (0,0,1)\}, x = (5, 10, 15)$

(4) Use the Gram-Schmidt orthonormalization process to transform the given basis of R^n into an orthonormal
basis.Use the Euclidean inner product for R^n and use the vectors in the order in which they are given.

(i) $B = \{(3,4), (1,0)\}$

(ii) $B = \{(4,-3,0), (1,2,0), (0,0,4)\}$

(5) Use the Gram-Schmidt orthonorm;ization process to transform the given basis of a subspace of R^n into an
orthonormal basis for thesubspace. Use the Euclidean inner product for R^n and use the vectors in the order in
which they are given

(i) $B = \{(3,4,0), (1,0,0)\}$

(ii) $B = \{(1,2,-1,0), (2,2,0,1)\}$

(6) Find an orthonormal basis for the solution space of the given homogeneous system of linear equations.

(i)
$$\begin{aligned} 2x_1 + x_2 - 6x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 - 3x_3 + 4x_4 &= 0 \\ x_1 + x_2 - 3x_3 + 2x_4 &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 + x_2 - x_3 - x_4 &= 0 \\ 2x_1 + x_2 - 2x_3 - 2x_4 &= 0 \end{aligned}$$

(7) Let $p(x) = a_0 + a_1 x - a_2 x^2$ and $q(x) = b_0 + b_1 x - b_2 x^2$ be vectors in P_2 with $\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$.
Determine whether the given second - degree polynomials form an orthonormal set, and if not, use the Gram - Schmidt orthonormalization process to form an orthonormal set.

(i) $\left\{ \frac{x^2+1}{\sqrt{2}}, \frac{x^2+x-1}{\sqrt{3}} \right\}$

(ii) $\{x^2, x^2 + 2x, x^2 + 2x + 1\}$

(8) Use the inner product $\langle p, q\rangle = a_0 b_0 + a_1 b_1 + a_2 b_2$. and the Gram - Schmidt orthormalization process to transform $\{(2,-1),(-2,10)\}$ into an orthonormal basis.

(9) Find an orthonormal basis for R^4 the includes the vectors

$$v_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0\right) \text{ and } v_2 = \left(0, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

(10) In each part an orthonormal basis relative to the Euclidean inner product in given . Find the coordinal vector of w with respect to that basis .

a) $w = (3,7)$, $u_1 = (\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}})$, $u_2 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$.

b) $w = (-1,0,2)$, $u_1 = (\frac{2}{3}, \frac{-2}{3}, \frac{1}{3})$, $u_2 = (\frac{2}{3}, \frac{1}{3}, \frac{-2}{3})$, $u_3 = (\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$

(11) Let R^2, have a Euclidean inner product and let $S = \{w1, w2\}$ be the orthonormal basis with $w_1 = (\frac{3}{5}, \frac{-4}{5})$, $w_2 = (\frac{4}{5}, \frac{3}{5})$.

a) Find the vector u and v that have coordinal vectors $(u)_S = (1,1)$ and $(v)_S = (-1,4)$.

b) Compute $\|u\|$, $d(u,v)$ and $\langle u, v\rangle$ to the coordinate vectors $(u)_S$ and $(v)_S$ then check the result by performing the computation directly on u, v.

(12) the subspace w of R^3, spanned by the vector $u_1 = (\frac{4}{5}, 0, \frac{-3}{5})$ and $u_2 = (0,1,0)$ is a plane porsing through the origin.

Express $u = (1,2,3)$ in the form $u = w_1 + w_2$, where w_1 lies in the plane and w_2 is perpendicular to the plane .

(13) Find the $QR-$ decompoition of the matrix :

a) $\begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix}$, b) $\begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 4 & 1 \end{pmatrix}$, c) $\begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 1 \end{pmatrix}$

Chapter III

Linear transformation

Definition *3.1.1:*

If $T: V \to W$ is a function from a vector space V into a vector space W,
then T is called a linear transformation from V to W if for all vectors u and v in V and all scalars c

(a) $T(u+v) = T(u) + T(v)$

(b) $T(cu) = cT(u)$

In the special case where $V = W$, the linear transformation $T: V \to V$ is called a linear operator on V.

Definition *3.1.2:*

If $T: V \to W$ is a linear transformation then the set of vectors in V that T maps into 0 is called the kernal of T and denoted by $ker(T)$.

The set of all vectors in W that are images under T of at least one vector in V is called the range of T denoted by $R(T)$.

Definition *3.1.3:*

If $T: V \to W$ is a linear transformation then the dimension of the range of T is called the rank of T
denoted by $rank(T)$ and the dimension of the kernal is called the nullity of T
denoted by $nullity(T)$.

Theorem *3.1.4:*

Dimension Theorem for linear transformation:

If $T: V \to W$ is a linear transformation from an n - dimensional vector space V to a vector space , then

$$rank(T) + nullity(T) = n = \dim \quad \text{domain}$$

3.2: Matrices of general linear transformation

In this section we shall show that if V and W are finite - dimensional vector spaces (not necessarily R^n and R^m),
then with a little ingenuity any linear transformation $T: V \to W$ can be regarded as a matrix transformation.

The basic idea is to work with coordinate matrices of the vectors rather than with the vectors themselves.

Matrices of linear transformation:

Suppose that V is an n - dimensional vector spase and W an m - dimensional vector spase.

If we choose bases B and $B^{'}$ for V and W, respectively, then for each x in V, the coordinate matrix $[x]_B$ will be A vector in R^n, and the coordinate matrix $[T(x)]_{B'}$ will be a vector in R^m (Figure 1).

If, as illustrated in Figure 2, we complete the rectangle suggested by Figure 1, we obtain a mapping from R^n to R^m, which can be shown to be a linear transformation. If we let A be the standard matrix for this transformation, then

$$A[x]_B = [T(x)]_{B'} \qquad (1)$$

The matrix A in (1) is called the matrix for T with respect to th bases B and $B^{'}$

Later in this section, we shall give some of the uses of the matrix A in (1), but first, let us show how it can be computed.

For this purpose, let us suppose that $B = \{u_1, u_2, \ldots, u_n\}$, is basis for the n -

dimensional space

V and $B' = \{v_1, v_2 \ldots, v_m\}$ is a basis for the m - dimensional space W,

We are looking for an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

such that (1) holds for all vectors x in V.

In particular, we want this equation to hold for the basis vectors $u_1, u_2, \ldots, u_n$, that is,

$$A[u_1]_B = [T(u_1)]_{B'},\ \ A[u_2]_B = [T(u_2)]_{B'}, \ldots, A[u_n]_B = [T(u_n)]_{B'} \qquad (2)$$

But

$$[u_1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, [u_2]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots\ldots, [u_n]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

so

$$A[u_1]_B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$$

$$A[u_2]_B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}$$

$$\vdots$$

$$A[u_n]_B = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}$$

Substituting these results into (2) yields

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} = [T(u_1)]_{B'}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} = [T(u_2)]_{B'}, \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = [T(u_n)]_{B'}$$

which shows that the successtive columns of A are the coordinate matrices of

$$T(u_1), T(u_2), \dots\dots\dots\dots, T(u_n)$$

with respect to the basis B'. Thus, the matrix for T with respect to the bases B and B' is

$$A = \left[[T(u_1)]_{B'} \vdots [T(u_2)]_{B'} \vdots \dots \vdots [T(u_n)]_{B'} \right] \tag{3}$$

This matrix is commonly denoted by the symbol

$$[T]_{B',B}$$

so that the preceding formula can also be written as

$$[T]_{B',B} = \left[[T(u_1)]_{B'} \vdots [T(u_2)]_{B'} \vdots \dots \vdots [T(u_n)]_{B'} \right] \tag{4}$$

and from (1) this matrix has the property

$$[T]_{B',B}[x]_B = [T(x)]_{B'} \tag{4a}$$

Remark *:*

Observe that in the notation $[T]_{B',B}$ the right subscript is a basis for the domain of T, and the left subscript is a basis for the image space of T (Figure 3)

Moreover, observe how the subscript B seems to " cancel out" in Formula (4a) (Figure 4)

Matrices of linear operators:

In the special case where $V = W$ (so that $T: V \to V$ is a linear operator) it is usual to take $B = B'$
when constructing a matrix for T. In this case the resulting matrix is called the matrix for T
with respect to the basis B and is usually denoted by $[T]_B$ rather than $[T]_{B,B}$.
If $B = \{u_1, u_2, \ldots, u_n\}$, then in this case Formulas (4) and (4a) become

$$[T]_B = \left[[T(u_1)]_B \vdots [T(u_2)]_B \vdots \ldots \vdots [T(u_n)]_B \right] \tag{5}$$

and

$$[T]_B[x]_B = [T(x)]_B \tag{5a}$$

Phrased informally, (4a) and (5a) state that the matrix for T times the coordinate matrix for x is the coordinate matrix for T(x).

Example *(1):*

Let $T: P_1 \to P_2$ be the linear transformation defined by

$T(p(x)) = xp(x)$
Find the matrix for T with respect to the standard bases
$B = \{u_1, u_2\}$ and $B' = \{v_1, v_2, v_3\}$
where ,
$u_1 = 1, \quad u_2 = x; \quad v_1 = 1, v_2 = x, v_3 = x^2$

Solution *:*

From the given formula for T we obtain
$T(u_1) = T(1) = x(1) = x$
$T(u_2) = T(1) = x(x) = x^2$
By inspection, we can determine the coordinate matrices for $T(u_1)$ and $T(u_2)$ relative to B';
they are

$$[T(u_1)]_{B'} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ [T(u_2)]_{B'} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, the matrix for T with respect to B and B' is

$$[T]_{B',B} = [[T(u_1)]_{B'} \vdots [T(u_2)]_{B'}] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Example *(2):*

Let $T: P_1 \to P_2$ be the linear transformation in Example (1). Show that the matrix

$$[T(u_1)]_{B',B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(obtaind in Example (1)) satisfies (4a) for every vector $x = a + bx$ in P_1

Solution *:*

Since $x = p(x) = a + bx$ we have
$T(x) = xp(x) = ax + bx^2$
For bases B and B' in Example (1). it follows that

$$[x]_B = [a + bx]_B = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$[T(x)]_{B'} = [ax + bx^2]_{B'} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix}$$

Thus

$$[T]_{B',B}[x]_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ a \\ b \end{bmatrix} = [T(x)]_{B'}$$

Example *(3):*

Let $T : R^2 \to R^3$ be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -5x_1 + 13x_2 \\ -7x_1 + 16x_2 \end{pmatrix}$$

Find the matrix for the transformation T with respect to the bases $B = \{u_1, u_2\}$ for R^2 and $B' = \{v_1, v_2, v_3\}$ for R^3, where

$$u_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}; \quad v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, v_2 = \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

Solution *:*

$$T(u_1) = T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix}$$

$$T(u_2) = T\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$T(u_1) = \begin{pmatrix} 1 \\ -2 \\ -5 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + b\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + c\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$a = 1, \quad b = 0, \quad c = -2$$

$$[T(u_1)]_{B'} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$T(u_1) = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} = d\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + e\begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + f\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$[T(u_1)]_{B'} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$$

$$[T]_{B',B} = \left[[T(u_1)]_B \vdots [T(u_2)]_B\right] = \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ -2 & -1 \end{bmatrix}$$

Example *(4):*

Let $T: R^2 \to R^2$ be the linear operator defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix}$$

and let $B = \{u_1, u_2\}$ be the basis, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

(a) Find $[T(u_1)]_B$

(b) Verify that 5a holds for every vector x in R^2.

Solution *:*

(a) $T(u_1) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2u_1 + 0u_2, \qquad T(u_2) = \begin{pmatrix} 3 \\ 6 \end{pmatrix} = 3u_2$

Therefore

$$[T(u_1)]_B = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \text{ and } [T(u_2)]_B = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$$[T]_B = \left[[T(u_1)]_B \vdots [T(u_2)]_B\right] = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

(b) $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad (1)$

is any vector in R^2, then from the given formula for T

$$T(x) = \begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix} \qquad (2)$$

To find $[x]_B$, $[T(x)]_B$ we must express (1) and (2) as a L.C. of u_1, u_2 so

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = k_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + k_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix};$$

$$\begin{pmatrix} x_1 + x_2 \\ -2x_1 + 4x_2 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

So

$$\begin{aligned} k_1 + k_2 &= x_1 \\ k_1 + 2k_2 &= x_2 \end{aligned} \qquad (3)$$

and

$$\begin{aligned} c_1 + c_2 &= x_1 + x_2 \\ c_1 + 2c_2 &= -2x_1 + 4x_2 \end{aligned} \qquad (4)$$

Solving (3) for k_1, k_2 we get

$k_1 = 2x_1 - x_2$

$k_2 = -x_1 + +x_2$

So

$$[x]_B = \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix}$$

Solving (4) for c_1, c_2 , yields

$c_1 = 4x_1 - 2x_2$

$c_2 = -3x_1 + 3x_2$

So that

$$[T(x)]_B = \begin{pmatrix} 4x_1 - 2x_2 \\ -3x_1 + 3x_2 \end{pmatrix}$$

Thus

$$[T]_B [x]_B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2x_1 - x_2 \\ -x_1 + x_2 \end{pmatrix}$$

$$\begin{pmatrix} 4x_1 - 2x_2 \\ -3x_1 + 3x_2 \end{pmatrix} = [T(x)]_B$$

Theorem *3.2.1:*

If $T : R^n \to R^m$ is a linear transformation and if B and B' are the standard bases for R^n and R^m respectively, then

$$[T]_{B',B} = [T]$$

This theorem tells a special case where T maps R^n into R^m ,
the matrix for T with respect to the standard basis is the standard matrix for T.
In this special case formula (4a) of this section reduces to

$$[T]_x = T(x)$$

To focus an the later idea:
Let $T: V \to W$ be a linear transformation, the matrix $[T]_{B',B}$ can be used to calculate $T(x)$ in three steps by indirect procedure.

x direct $\longrightarrow$ $T(x)$

(1) $\uparrow$ $\uparrow$ (3)

$[x]_B$ Multiply by $\longrightarrow$ $[T(x)]_{B'}$

$[T]_{B',B}$

(2)

(1) Compute the coordinate matrix $[x]_B$.
(2) Multiply $[x]_B$ an the left by $[T]_{B',B}$ to produce $[T(x)]_{B'}$
(3) Reconstruct $T(x)$ from its coordinate matrix $[T(x)]_{B'}$

Example *(5):*

Let $T: P_2 \to P_2$ be the linear operator defined by
$T(p(x)) = p(3x - 5)$
that is, $T(c_0 + c_1x + c_2x^2) = c_0 + c_1(3x - 5) + c_2(3x - 5)^2$

(a) Find $[T]_B$ with respect to the basis $B = \{1, x, x^2\}$
(b) Use the indirect procedure to complete $T(1 + 2x + 3x^2)$
(c) Check the result in (b) by computing $T(1 + 2x + 3x^2)$ directily

Solution *:*

$$T(1) = 1, \quad T(x) = (3x - 5), \quad T(x^2) = (3x-5)^2 = 9x^2 - 30x + 25$$

$$[T(1)]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, [T(x)]_B = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}, [T(x^2)]_B = \begin{pmatrix} 25 \\ -30 \\ 9 \end{pmatrix},$$

Thus

$$[T]_B = \begin{pmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{pmatrix}$$

(b) The coordinate matrix relative to B for the vector $p = 1 + 2x + 3x^2$ is

$$[P]_B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Thus from (5a)

$$[T(1 + 2x + 3x^2)]_B = [T(p)]_B = [T]_B[P]_B$$

$$\begin{pmatrix} 1 & -5 & 25 \\ 0 & 3 & -30 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 66 \\ -84 \\ 27 \end{pmatrix}$$

from which it follows that $T(1 + 2x + 3x^2) = 66 - 84x + 27x^2$

c) By direct computation

$$\begin{aligned} T(1 + 2x + 3x^2) &= 1 + 2(3x - 5) + 3(3x + 5)^2 \\ &= 1 + 6x - 10 + 27x^2 - 90x + 75 \\ &= 66 - 84x + 27x^2 \end{aligned}$$

Problem Set V

(1) Let $T: P_2 \to P_3$ be the linear transformation defined by

$$T(p(x)) = xp(x)$$

(a) Find the matrix for T w.r.t. the standard basis
$B = \{u_1, u_2, u_3\}$ and $B' = \{v_1, v_2, v_3, v_4\}$

where
$u_1 = 1, \quad u_2 = x, \quad u_3 = x^2$;
$v_1 = 1, \quad v_2 = x, \quad v_3 = x^2, \quad v_4 = x^3$

(b) Verify that the matrix $[T]_{B',B}$ obtained in part (a) satisfies formula (4a) for every vector
$x = c_0 + c_1x + c_2x^2$ in P_2

(2) Let $T: P_2 \to P_2$ be the linear operator defined by
$T(a_0 + a_1x + a_2x^2) = a_0 + a_1(x-1) + a_2(x-1)^2$

(a) Find the matrix of T w.r.t the standard basis $B = \{1, x, x^2\}$ for P_2
(b) Verify that the matrix $[T]_B$ obtained in (z) satisfy formula (5a) for every vector

$$x = a_0 + a_1x + a_2x^2 \text{ in } P_2$$

(3) Let $T: R^2 \to R^3$ be defined by

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_2 \\ -x_1 \\ 0 \end{pmatrix}$$

(a) Find the matrix $[T]_{B',B}$ w.r.t. the bases $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2, v_3\}$, where

$$u_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -2 \\ 4 \end{pmatrix}; \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

(b) Verify that formula (4a) holds for every vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in R^2

(4) Let $T: P_2 \to P_2$ be the linear operator defined by

$$T(p(x)) = p(2x+1)$$

that is, $T(c_0 + c_1 x + c_2 x^2) = c_0 + c_1(2x+1) + c_2(2x+1)^2$

(a) Find $[T]_B$ with respect to the basis $B = \{1, x, x^2\}$
(b) Use the indirect procedure to complete $T(2 - 3x + 4x^2)$
(c) Check the result obtained in part (b) by computing $T(2 - 3x + 4x^2)$ directily

5) Let $v_1 = \begin{pmatrix} \frac{1}{3} \end{pmatrix}$ and $v_2 = \begin{pmatrix} \frac{-1}{4} \end{pmatrix}$ and let $A = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$ be the matrix for

$T: R^2 \to R^2$ w.r.t the basis $B = \{v_1, v_2\}$

a) Find $[T(v_1)]_B$ and $[T(v_2)]_B$

b) Find $T(v_1)$ and $T(v_2)$

c) Find a formula for $T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

d) Use the formula obtoined in (c) to compute $T\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Solution :

$$[T(v_1)]_B = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$[T(v_2)]_B = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$$

$$T(v_1) = 1\begin{pmatrix} 1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$$

$$T(v_2) = 3\begin{pmatrix} 1 \\ 3 \end{pmatrix} + 5\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 29 \end{pmatrix}$$

$$[T]_B[x]_B = [T(x)]_B$$

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = a\begin{pmatrix} 1 \\ 3 \end{pmatrix} + b\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 & x_1 \\ 3 & 4 & x_2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{4x_1+x_2}{7} \\ 0 & 1 & \frac{-3x_1+x_2}{7} \end{pmatrix}$$

$$[x]_B = \begin{pmatrix} a = \frac{4x_1+x_2}{7} \\ b = \frac{-3x_1+x_2}{7} \end{pmatrix}$$

$$A[x]_B = [T(x)]_B = \begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} \frac{4x_{1+x_2}}{7} \\ \frac{-3x_1+x_2}{7} \end{pmatrix}$$

$$= \begin{pmatrix} c = \frac{-5x_1+4x_2}{7} \\ d = \frac{-23x_1+3x_2}{7} \end{pmatrix}$$

$$T\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = c\begin{pmatrix} 3 \\ 1 \end{pmatrix} + d\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$= \frac{-5x_1+4x_2}{7}\begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{-23x_1+3x_2}{7}\begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-5x_1+4x_2+23x_1-3x_2}{7} \\ \frac{-15x_1+12x_2-92x_1+12x_2}{7} \end{pmatrix} = \begin{pmatrix} \frac{18x_1+x_2}{7} \\ \frac{-107x_1+24x_2}{7} \end{pmatrix}$$

$$\therefore \quad T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{19}{7} \\ \frac{-83}{7} \end{pmatrix}$$

3.3 Similarity :

We will see how the matrices of a linear transforation relative to two different bases are related we note that :

1- Matrix of T relation to B : A
2– Matrix of T relation to B' : A'
3- Transtion matrix from B' to B : P
4- Transtion matrix from B to B' : P'
we now show the relationship among A , A', P, P'
Tthis mean that

$$A'[v]_{B'} = [T(v)]_B$$

$$P^{-1}AP[v]_B = [T(v)]_B$$

this implis that

$$A' = P^{-1}AP$$

$$[v]_B = P[v]_B$$

$$[T(v)]_B = A[v]_B$$

$$[T(v)]_B = P^{-1}[T(v)]_B$$

Example *(6):*

Find the matrix A' for $T : R^2 \to R^2$

$$T(x_1, x_2) = (2x_1 - 2x_2, -x_1 + 3x_2)$$

relative to the basis $B' = \{(1,0),(1,1)\}$, $B = \{(1,0),(-2,3)\}$

Solution *:*

We find the standrad matrix for T

$$A = \begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}$$

there transition matrix from B' to the standrad basis $B = \{(1,0),(0,1)\}$ is
$(1,1) = 1(1,0) + 0(0,1)$
$(1,1) = 1(1,0) + 1(0,1)$
$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, P^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
therefore the matrix from T relatire to B' is

$$A' = P^{-1}AP = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ -1 & 2 \end{pmatrix}$$

Example *(7):*

Let $B = \{(-3,2),(4,-2)\}$ and $B' = \{(-1,2),(2,-2)\}$ be base for R^2,

let $A = \begin{pmatrix} -2 & 7 \\ -3 & 7 \end{pmatrix}$ be the matrix for $T, R^2 \to R^2$ relatire to B,

Find A', $[v]_B$,$[T(v)]_B$ and $[T(v)]_B$ for the coorditate matrix $[v]_{B'} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$

Solution *:*

$$A' = P^{-1}AP$$

to find P we note that

$(-1,2) = a(-3,2) + b(4,-2) \Rightarrow a = 3, b = 2$

similarty,

$(2,-2) = c(-3,2) + d(4,-2) \Rightarrow c = -2, d = -1$

$P = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$,the $P^{-1} = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}$ hence

$$A' = \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} -2 & 7 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$$

since

$[v]_{B'} = \begin{pmatrix} -3 \\ -1 \end{pmatrix}$,the

$$[v]_B = P[v]_{B'} = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ -5 \end{pmatrix}$$

$$[T(v)]_B = A[v]_B$$

$$= \begin{pmatrix} -2 & 7 \\ -3 & 7 \end{pmatrix}\begin{pmatrix} -7 \\ -5 \end{pmatrix} = \begin{pmatrix} -21 \\ -14 \end{pmatrix}$$

$$[T(v)]_{B'} = P^{-1}[T(v)]_B$$

$$= \begin{pmatrix} -1 & 2 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} -21 \\ -14 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix}$$

or by

$$[T(v)]_{B'} = A'[v]_{B'} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -3 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix}$$

Definition *3.3.1:*

For square matrices A and A' of order n, A' is said to be similer to A if there exists an invertible matrix P such that

$$A' = P^{-1}AP$$

Theorem *3.2.2* *(similarity an equivelance reletion)*

let A, B and C be square matrices of order n ,
Then the following properties are true
1) A is similer to A (reflexive)
2) If A is similer to B ,then B is similar to A (symmetric)
3) If A is similer to B and B is similar to C,then A (transitive)

Proof :

1)The first property follows from the fact that

$$A = I_n^{-1}AI_n$$
$$= I_nAI_n$$

2) $A = P^{-1}BP$

$$PAP^{-1} = (PP^{-1})B(PP^{-1}) = B$$

Let $Q = P^{-1}$
so $Q^{-1}AQ = B$

3) Let $A = P^{-1}AP, \quad B = Q^{-1}CQ$

$$\begin{aligned} A &= P^{-1}(Q^{-1}CQ)P \\ &= (P^{-1}Q^{-1})C(QP) \\ &= (QP)^{-1}C(QP) \qquad \text{let } R = QR \\ &= R^{-1}CR \end{aligned}$$

so A is similer to C
$\therefore$ Similrity is an equivelance relation

TABLE 1:Similarty Invariants

Property	Description
Determinant	A and $P^{-1}AP$ have the same determinant
Invertibility	A is invertible if and only if $P^{-1}AP$ is invertimle
Rank	A and $P^{-1}AP$ have the same rank
Nullity	A and $P^{-1}AP$ have the same nullity
Trace	A and $P^{-1}AP$ have the same trace
Characteristic polynomial	A and $P^{-1}AP$ have the same characteristic polynomial
Eigenvalues	A and $P^{-1}AP$ have the same eigenvalues
Eigenspace dimension	If λ is an eigenvalue of A and $P^{-1}AP$,then the eigenspace of A corresponding to λ and the eigenspace of $P^{-1}AP$ corresponding to λ have the same dimension.

Problem Set VI

(1) (a) Find the matrix A' for T relative to the basis B' and
(b) Show that A' is similar to A, the standard matrix for T.

(i) $T: R^3 \to R^3, \quad T(x, y, z) = (x, y, z)$.
$B' = \{(1,1,0), (1,0,1), (0,1,1)\}$.

(ii) $T: R^3 \to R^3, \quad T(x, y, z) = (x - y + 2z, 2x + y - z, x + 2y + z)$.
$B' = \{(1,0,1), (0,2,2), (1,2,0)\}$.

(2) Let $B = \{(1,1,0), (1,0,1), (0,1,1)\}$ and
$B' = \{(1,0,0), (0,1,0), (0,0,1)\}$ be basis for R^3 ,and let

$$A = \begin{bmatrix} \frac{3}{2} & -1 & \frac{-1}{2} \\ \frac{-1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{5}{2} \end{bmatrix}$$

be the matrix for $T: R^3 \to R^3$ relative to B.
(a)Find the transition matrix P from B' to B.
(b)Use the matrices A and P to find $[v]_B$ and $[T(v)]_{B'}$, where

$$[v]_{B'} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

(c) Find A' (the matrix of T relative to B') and P^{-1}.
(d) Find $[T(v)]_{B'}$ in two ways: first as $P^{-1}[T(v)]_B$ and then as $A'[v]_{B'}$

(3) Prove that if A and B are similar, then $|A| = |B|$

(4) Prove that if A is similar to B and B is similar to C, then A is similer to C.

(5) Prove that if A and B are similer, then A^2 is similer to B^2.

(6) Let $A = CD$, where C is an invertible $n \times n$ matrix, Prove that the matrix DC is similer to A.

CHAPTER IV

Eigenvalues and Eigenvectors

4.1:Eigenvalues and Eigenvectors:

Here we introduce one of the most important problems in Linear Algebra,called the eigenvalue problem.

Definition *4.1.1:*

Let A be an $n \times n$ matrix ,the scalar λ is called the eigenvalue of A if there is a nonzero vector x such that

$$Ax = \lambda x$$

The vector x is called the eigenvector of A corresponding to λ.

Note:

eigenvectors can not be zero.

Theorem *4.1.2:*

If A is an $n \times n$ matrix and λ is a real number ,then the following are equivalent :

a) λ is an eigenvalue of A.

b)The system of equation $(\lambda I - A)x = 0$ has nontrivial solution .

c) There is a nonzero vector x in R^n such that $Ax = \lambda x$.

d) λ is a solution of the characteristic equation $det(\lambda I - A) = 0$.

Example *(1):*

Find the eigenvalues and the basis for the eigenspace of $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & 10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{pmatrix}$

Solution *:*

The characteristic equation of A is :

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 0 & 0 & 0 \\ 0 & \lambda - 1 & -5 & -10 \\ -1 & 0 & \lambda - 2 & 0 \\ -1 & 0 & 0 & \lambda - 3 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2)(\lambda - 3) = 0$$

Thus the eigenvalues are $\lambda = 1, 2, 3$

If $\lambda = 1$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & -10 \\ -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$x_{1+2}x_4 = 0 \Rightarrow x_1 = -2x_4 = -2t$
Let $x_4 = t$
$x_3 - 2x_4 = 0 \Rightarrow x_3 = 2x_4 = 2t$
$x_2 = s$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2t \\ s \\ 2t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 0 \\ 2 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

So the basis of the eigenspace corresponding to $\lambda_1 = 1$ is $B_1 = \{(0,1,0,0),(-2,0,2,1)\}$
the basis of the eigenspace corresponding to $\lambda_2 = 2$ is $B_2 = \{(0,5,1,0)\}$
the basis of the eigenspace corresponding to $\lambda_3 = 3$ is $B_3 = \{(0,-5,0,1)\}$

4.2 :The Theorems of cayley-Hamilton .

There are many interesting result concerning the eigenvalues of a matrix .It says that any matrix satisfies its own characteristic equation .

Let

$$P(x) = x^n + a_{n-1}x^{n-1} + \ldots\ldots + a_1 x + a_0$$

be a polynomial and let A be an $n \times n$ matrix .Then power of A are defined and we define :

$$P(A) = A^n a_{n-1} + A^{n-1} + \ldots\ldots\ldots + a_1 A + a_0 I_n \qquad (1)$$

Example *(2) :*

Let $A = \begin{pmatrix} -1 & 4 \\ 3 & 7 \end{pmatrix}$ and $P(x) = x^2 - 5x + 3$.

Then $P(A) = A^2 - 5A + 3I_n$

$$= \begin{pmatrix} -1 & 4 \\ 3 & 7 \end{pmatrix}\begin{pmatrix} -1 & 4 \\ 3 & 7 \end{pmatrix} - 5\begin{pmatrix} -1 & 4 \\ 3 & 7 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 13 & 24 \\ 18 & 61 \end{pmatrix} + \begin{pmatrix} 5 & -20 \\ -15 & -35 \end{pmatrix} + \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 21 & 4 \\ 3 & 29 \end{pmatrix}$$

(1) is polynomial with scalar coefficient defined for a matrix variable .We can also define a polynomial with square matrix coefficient by :

$$Q(\lambda) = B_0 + B_1\lambda + B_2\lambda^2 + \ldots\ldots + B_n\lambda^n \qquad (2)$$

If A is a matrix ,then we define :

$$Q(A) = B_0 + B_1A + B_2A^2 + \ldots\ldots + B_mA^m \qquad (3)$$

we must be carful in (3) since matrices do not commut under multiplication .

Theorem *4.2.1:*

If $P(\lambda)$ and $Q(\lambda)$ are polynomials in the scalar variable λ with square matrix coefficients and if $P(\lambda) = Q(\lambda)(A - \lambda I)$,then

$$P(A) = 0$$

Proof :

If $Q(\lambda)$ is given by equation (2) ,then

$$P(\lambda) = (B_0 + B_1\lambda + B_2\lambda^2 + \ldots\ldots + B_n\lambda^n)(A - \lambda I).$$

$$= B_0A + B_1A\lambda + B_2A\lambda^2 + \ldots\ldots + B_nA\lambda^n - B_0\lambda - B_1\lambda^2 - B_2\lambda^3 - \ldots\ldots - B_n\lambda^{n+1} \qquad (4)$$

Then substituting A for λ in (4) we obtain

$$P(A) = B_0A + B_1A^2 + B_2A^3 + \ldots\ldots + B_nA^{n+1} - B_0A - B_1A^2 - B_2A^3 - \ldots\ldots - B_nA^{n+1} = 0$$

Theorem *4.2.2 : (The cayley-Hamilton Theorem):*

Every square matrix satisfies its own characteristic equation .That is ,If $P(\lambda) = 0$ is the characteristic equation of A , then

$$P(A) = 0$$

Proof :

We have

$$P(\lambda) = det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & \ldots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} - \lambda \end{vmatrix}$$

any cofactor of $(A - \lambda I)$ is a polynomial in λ .Thus the adjoint of $(A - \lambda I)$ is an $n \times n$

matrix each of whose component is a polynomial in λ .

That is

$$adj(A-\lambda I) = \begin{pmatrix} p_{11}(\lambda) & p_{12}(\lambda) & \ldots & p_{1n}(\lambda) \\ p_{21}(\lambda) & p_{22}(\lambda) & \ldots & p_{2n}(\lambda) \\ \vdots & \vdots & \vdots & \vdots \\ p_{n1}(\lambda) & p_{n2}(\lambda) & \ldots & p_{nn}(\lambda) \end{pmatrix}$$

This means that we can think of $adj(A-\lambda I)$ as a polynomial $Q(\lambda)$, in λ with $n \times n$ matrix coefficients .

To see this look at the following :

$$\begin{pmatrix} -\lambda^2-2\lambda+1 & 2\lambda^2-7\lambda-4 \\ 4\lambda^2+5\lambda-2 & -3\lambda^2-\lambda+3 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ 4 & -3 \end{pmatrix}\lambda^2 + \begin{pmatrix} -2 & -7 \\ 5 & -1 \end{pmatrix}\lambda + \begin{pmatrix} 1 & -4 \\ -2 & 3 \end{pmatrix}$$

by Theorem let A be an $n \times n$ matrix .Then

$$(adjA)A = \begin{pmatrix} detA & 0 & 0 & \ldots & 0 \\ 0 & detA & 0 & \ldots & 0 \\ 0 & 0 & detA & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & detA \end{pmatrix} = det(A)I.$$

$$P(\lambda)I = det(A-\lambda I)I = [adj(A-\lambda I)][A-\lambda I] = Q(\lambda)(A-\lambda I) \qquad (5)$$

But

$$det(A-\lambda I)I = P(\lambda)I$$

. if

$$p(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots\ldots\ldots + a_1\lambda + a_0$$

then we define

$$P(\lambda) = P(\lambda)I = \lambda^n I + a_{n-1}\lambda^{n-1} + \ldots\ldots\ldots + a_1\lambda I + a_0 I.$$

Thus from (5) we have :

$$P(\lambda) = Q(\lambda)(A-\lambda I)$$

Finally ,from Theorem 4.2.1

$$P(A) = 0$$

.

Example *(3):*

Let $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$

$$\therefore\ |\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & -4 \\ -3 & \lambda - 2 & 1 \\ -2 & -1 & \lambda + 1 \end{vmatrix}$$

$\Rightarrow\ (\lambda - 1)[(\lambda^2 - \lambda - 1)] - [(-3\lambda - 3 + 2)] - 4[3 + 2\lambda - 4] = 0$
$\Rightarrow\ (\lambda - 1)(\lambda^2 - \lambda - 1) + 3\lambda + 1 + 4 - 8\lambda = 0$
$\Rightarrow\ \lambda^3 - 2\lambda^2 - \lambda + \lambda + 1 + 3\lambda + 5 - 8\lambda = 0$
$\Rightarrow\ \lambda^3 - 2\lambda^2 - 5\lambda + 6 = 0$

Now , we compute

$$A^2 = \begin{pmatrix} 6 & 1 & 1 \\ 7 & 0 & 11 \\ 3 & -1 & 8 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix}$$

and

$$A^3 - 2A^2 - 5A + 6I = \begin{pmatrix} 11 & -3 & 22 \\ 29 & 4 & 17 \\ 16 & 3 & 5 \end{pmatrix} + \begin{pmatrix} -12 & -2 & -2 \\ -14 & 0 & -22 \\ -6 & 2 & -16 \end{pmatrix} + \begin{pmatrix} -5 & 5 & -20 \\ -15 & -10 & 5 \\ -10 & -5 & 5 \end{pmatrix}$$

In same situation the Caylely Hamilton theorem is useful in calculating the inverse of a matrix . if A^{-1} exist and $P(A) = 0$, then

$$A^{-1}P(A) = 0$$

To illustrate , if

$$P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots\ldots\ldots + a_1\lambda + a_0$$

then

$$P(A) = A^n + a_{n-1}A^{n-1} + \ldots\ldots + a_1 A + a_0 I = 0$$

and

$$A^{-1}P(A) = A^{n-1} + a_{n-1}A^{n-2} + \ldots\ldots\ldots + a_2 A + a_1 I + a_0 A^{-1} = 0$$

Thues

$$A^{-1} = \frac{1}{a_0}(-A^{n-1} - a_{n-1}A^{n-2} - \ldots\ldots\ldots - a_2 A - a_1 I)$$

Note that $a_0 \neq 0$ because $a_0 = \det(A)$ and we assumed that A was invertible.

Example *(4):*

Let $A = \begin{pmatrix} 1 & -1 & 4 \\ 3 & 2 & -1 \\ 2 & 1 & -1 \end{pmatrix}$..

Solution *:*

Then $p(\lambda) = \lambda^3 - 2\lambda^2 - 5\lambda + 6$
Here $n = 3$, $a_2 = -2$, $a_1 = -5$, $a_0 = 6$

$$A^{-1} = \tfrac{1}{6}(-A^2 + 2A + 5I)$$

$$= \tfrac{1}{6}\left[\begin{pmatrix} -6 & -1 & -1 \\ -7 & 0 & -11 \\ -3 & 1 & -8 \end{pmatrix} + \begin{pmatrix} 2 & -2 & 8 \\ 6 & 4 & -2 \\ 4 & 2 & -2 \end{pmatrix} + \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}\right] = \tfrac{1}{6}\left[\begin{pmatrix} 1 & -3 & 7 \\ -1 & 9 & -13 \\ 1 & 3 & -5 \end{pmatrix}\right.$$

(4.3) : Eigenvalues of the powers of a matrix:-

Once the eigenvalues and eigenvectors of a matrix A are found its simple to find the eigenvalues
and the eigenvectors of any positive integer power of A , for example if λ is an eigenvalue of A
and x is a corresponding eigenvectors , then

$$A^2x = A(Ax) = A(\lambda x) = \lambda(Ax) = \lambda(\lambda x) = \lambda^2 x$$

which show that λ^2 is an eigenvalue of A^2 and x is a corresponding eigenvectors.

Theorem *4.3.1:*

If k is a positive integer , λ is an eigenvalue of a matrix A , and x is a corresponding eigenvectors , then λ^k is an eigenvalue of A^k and x is a corresponding eigenvector.

Example *(5):*

If $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

we have $\lambda = 1$, $\lambda = 2$
from theorem $\lambda = 2^7 = 128$ and $\lambda = 1^7 = 1$ are eigenvalues of A^7
we also have

$$x = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

are eigenvectors of A corresponding to the eigenvalue $\lambda = 2$
they are also eigenvectors of A^7 corresponding to $\lambda = 2^7 = 128$

Similarly , the eigenvector $\begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ of A corresponding to the eigenvalue $\lambda = 1$

is also an eigenvector of A^7 corresponding to $\lambda = 1^7 = 1$.

Remark *:-*

If an eigenvalue λ_1 occurs as a multiple root (k times) for the characteristic poly. we say that λ_1 has multiplicity k, The multiplicity of an eigenvalue is greater than or equal to the dimension of its eigenspace.

Example *(6):*

Find the eigenvalues and the corresponding eigenvectors for $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

Solution *:*

$$|\lambda I - A| = \begin{vmatrix} \lambda - 2 & -1 & 0 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 2)[(\lambda - 2)^2] = 0 \Rightarrow \lambda = 2$$

$(2I - A)x = 0$

$$\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow x_2 = 0 \,,\ x_3 = s \,,\ x_1 = t \,,\ s, t \text{ are not both zero}$$

$$x = \begin{pmatrix} t \\ 0 \\ s \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$, dim of eigenspace =2.

PROBLEM SET VII

(1) Verify that λ , is an eigenvalue of A and that x_i is a corresponding eigenvector.

(i) $A = \begin{bmatrix} 1 & k \\ 0 & -1 \end{bmatrix}$, $\lambda_1 = 1$, $x_1 = (1,0)$

(ii) $A = \begin{bmatrix} 4 & -5 \\ 2 & -3 \end{bmatrix}$, $\lambda_1 = -1, x_1 = (1,1)$
$\lambda_2 = 2, x_2 = (5,2)$

(iii) $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$, $\lambda_1 = 5, x_1 = (1,2,-1)$
$\lambda_2 = -3, x_2 = (-2,1,0)$
$\lambda_3 = -3, x_3 = (3,0,1)$

(2) Determine whether x is an eigenvector of A.

$A = \begin{bmatrix} -1 & -1 & 1 \\ -2 & 0 & -2 \\ 3 & -3 & 1 \end{bmatrix}$,

a) $x = (2,-4,6)$
b) $x = (2,0,6)$
c) $x = (2,2,0)$
d) $x = (-1,0,1)$

(3) Find:-

(a) the characteristic equation and (b) the eigenvalues (and corresponding eigenvectors) of the matrix.

(i) $\begin{bmatrix} 6 & -3 \\ -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

(4) Demonstrate the Cayley-Hamilton Theorem for the given matrix .
The Cayley-Hamilton Theorem states that a matrix satisfies its characteristic equation . For example,the characteristic equation

of $A = \begin{bmatrix} 1 & -3 \\ 2 & 5 \end{bmatrix}$ is $\lambda^2 - 6\lambda + 11 = 0$,

and therefore , by the theorem , we have $A^2 - 6A + 11I_2 = 0$

(i) $\begin{bmatrix} 4 & 0 \\ -3 & 2 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -4 \\ 0 & 3 & 1 \\ 2 & 0 & 1 \end{bmatrix}$

(5) For an invertible matrix A ,prove that A and A^{-1} have the same eigenvectors.
How are the eigenvalues of A related to the eigenvalues of A^{-1} ?

(6) If the eigenvalues of $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ are $\lambda_1 = 0$ and $\lambda_2 = 1$,
what are the possible values of a and d?

(7) Find the dimension of the eigenspace corresponding to the eigenvalue 3.

(i) $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(8) a) Find the characteristic equation $p(\lambda) = 0$ of the given matrix.
b) Verify that $p(A) = 0$.
c) Use part (b) to compute A^{-1}.

(i) $\begin{pmatrix} -2 & -2 \\ -5 & 1 \end{pmatrix}$ (ii) $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$

(9) Using the Cayley-Hamilton theorem compute A^{-1} of

$$A = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \\ -2 & -1 & 4 \end{pmatrix}.$$

4.4:Diogonalization :

Definition *4.4.2:*

An $n \times n$ matrix A is diagonalizable if A is similar to a diagonal matrix D. That is,A is diagonalizable

if there exists an invertible matrix P suth that

$$P^{-1}AP = D$$

is a diagonal matrix.

Note:

Every diagonal matrix D is diagonolizable since the identity matrix can play the role

of P to give $D = I^{-1}DI$

Theorem *4.4.2:*

If A and B are similar $n \times n$ matrices ,then they have the same eigenvalues.

Proof :

since A and B are similar ,there exists an invertible matrix P such that

$$B = P^{-1}AP$$

By properties of determinants

$$\begin{aligned}
|\lambda I - B| &= |\lambda I - P^{-1}AP| \\
&= |P^{-1}\lambda IP - P^{-1}AP| \\
&= |P^{-1}(\lambda I - A)P| \\
&= |P^{-1}||\lambda I - A||P| \\
&= |P^{-1}||P||\lambda I - A| \\
&= |P^{-1}P||\lambda I - A| \\
&= |\lambda I - A|
\end{aligned}$$

This means that A and B have the same characteristic polynomial. Hence they must have the same eigenvalues.

Example *(7):*

The following matrices A and D are similar

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & -2 & 4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Find the eigenvalues of A and D ?

Solution *:*

Since D is a diagonal matrix , then its eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

since A is similar to B then A has the same eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

Theorem *4.4.3*

An $n \times n$ matrix A diagonalizable if and it has n linearly independent eigenvectors

Proof .

First we assume that A is diagonalizable ,then there existe an invertible matrix P such that $P^{-1}AP = D$ is diagonal
letting the main diagonal entries of D be λ_1 , $\lambda_2 \ldots\ldots\ldots \lambda_n$ and the column vectors of P be $P_1, P_2, \ldots, P_n$ produces

$$PD = [P_1 P_2 \ldots\ldots P_n] \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = [\lambda_1 p_1 \vdots \lambda_2 p_2 \vdots \ldots\ldots\ldots \vdots \lambda_n p_n]$$

but since

$$AP = [AP_1 \vdots AP_2 . \vdots \ldots\ldots \vdots . AP_n]$$

and $P^{-1}AP = D$ we have $AP = PD$ which implies that

$$[AP_1 \vdots AP_2 \vdots \ldots\ldots \vdots . AP_n] = [\lambda_1 P_1 \vdots \lambda_2 P_2 \vdots \ldots\ldots\ldots \vdots \lambda_n P_n]$$

In other words , $AP_i = \lambda_i P_i$ for each column vector
This means that the column vectors P_i of P are eigenvectors of A
since P is invertible ,it column vectors are linearly indepent .This A has n linearly independent eigenvectors
Conversly: assume A has n linearly independent eigenvectors $P_1, P_2, \ldots\ldots\ldots, P_n$ with corresponding
let P be the matrix whose colums are these n-eigenvectors ,that is $P = [P_1 \vdots P_2 \vdots \ldots\ldots \vdots P_n]$
since each P_i is an eigenvectors of A ,we have $AP_i = \lambda_i P_i$ and

$$AP = A[P_1 \vdots P_2 \vdots \ldots\ldots \vdots P_n] = [\lambda_1 P_1 \lambda_2 p_2 \ldots\ldots\ldots\ldots \lambda_n P_n]$$

The right hand matrix can be written as follows

$$AP = \left[P_1 \; P_2 \ldots \vdots . P_n\right] \begin{bmatrix} \lambda_1 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2 & \ldots\ldots & 0 \\ \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix} = PD$$

since the vectors $P_1, P_2, \ldots, P_n$ are linearly independent , P is invertible and we write $AP = PD$
as $P^{-1}AP = D$ that is A is diogonalizable

Steps for Diagonalizing an $n \times n$ Square Matrix

Let A be an $n \times n$ matrix.

1- Find n linearly independent eigenvectors $p_1, p_2, \ldots, p_n$ for A,with corresponding eigenvalues $\lambda_1, \lambda_2, \ldots. \lambda_n$ .If n linearly independent eigenvectors do not exist ,then A is not diagonalizable .

2-If A has n linearly independent eigenvectors ,let P be the $n \times n$ matrix whose columns cosist of these eigenvectors .That is

$$P = [p_1 \vdots p_2 \vdots \ldots \vdots p_n].$$

3-The diagonal matrix $D = P^{-1}AP$ will have the eigenvalues $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ on its main diagonal (and zeros elsewhere) Note that the order of eigenvectors used to form P will determine the order in which the eigenvalues appear on the main diagonal of D.

Example *(8):A matrix that is not diagonalizable*

Show that the following matrix is not diagonalizable

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Solution *:*

Since A is triangular , the eigenvalues are simply the entries on the main diagonal . Thus the only eigenvalues is $\lambda = 1$. The matrix $(I - A)$ has the following reduced row-echelon form

$$I - A = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

this implies that $x_2 = 0$, and letting $x_1 = t$, we find that every eigenvector of A has the form

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence A does not have two linearly independent eigenvectors , and we conclude that A is not diagonalizable

Example *(9): Diagonalizing a Matrix*

Show that the following matrix is diagonalizable

$$A = \begin{bmatrix} 1 & -1 & -1 \\ 1 & 3 & 1 \\ -3 & 1 & -1 \end{bmatrix}$$

then find a matrix P such that $P^{-1}AP$ is diagonal

Solution :

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 1 & 1 \\ -1 & \lambda - 3 & -1 \\ 3 & -1 & \lambda + 1 \end{vmatrix} = (\lambda - 1)(\lambda + 2)(\lambda - 3) = 0$$

thus $\lambda_1 = 2$, $\lambda_2 = -2$, $\lambda_3 = 3$.

If $\lambda_1 = 2$

$$\therefore \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 3 & -1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -4 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3 = -t$

$x_2 = 0$, let $x_3 = t$

$$x = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

If $\lambda_2 = -2$

$$\therefore \begin{pmatrix} -3 & 1 & 1 \\ -1 & -5 & -1 \\ 3 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & \frac{-1}{4} \\ 0 & 1 & \frac{1}{4} \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 - \frac{-1}{4} x_3 = 0 \Rightarrow x_1 = \frac{4}{4} t = t.$

$x_2 + \frac{1}{4} x_3 = 0 \Rightarrow x_2 = -\frac{4}{4} t = -t.$

let $x_3 = 4t$.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t \\ -t \\ 4t \end{pmatrix} = t\begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

If $\lambda_3 = 3$

$$\therefore \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 3 & -1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

$x_1 + x_3 = 0 \Rightarrow x_1 = -x_3 = -t$
$x_2 - x_3 = 0 \Rightarrow x_2 = x_3 = t$
let $x_3 = t$.

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

test of p_1, p_2, p_3

$$\alpha_1 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + \alpha_3 \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\alpha_1 = \alpha_2 = \alpha_3 = 0$

$$P = \begin{pmatrix} -1 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 4 & 1 \end{pmatrix}$$

$$P^{-1} = \begin{pmatrix} -1 & -1 & 0 \\ \frac{1}{5} & 0 & \frac{1}{5} \\ \frac{1}{5} & 1 & \frac{1}{5} \end{pmatrix}$$

$$P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Theorem *4.4.4: (Sufficient condition for diagonalization)*

If an $n \times n$ matrix A has n distinct eigenvalues ,then the corresponding eigenvectors are linearly independent and A isdiagonalizable.

Proof :

let $\lambda_1, \lambda_2 \ldots\ldots\ldots \lambda_n$ be n distinct eigenvalues of A corresponding to the eigenvectors $x_1, x_2, \ldots.., x_n$,

to begin ,we assume that the set of eigenvectors is linearly dependent .Moreove,we consider the eigenvectors

to be ordered so that the first m eigenvectors are linearly independent ,but the first $m + 1$ are dependent ,where $m < n$.

then we can write x_{m+1} as alinear combination of the first m eigenvectors:

$$x_{m+1} = c_1x_1 + c_2x_2 + \ldots\ldots\ldots + c_mx_m \qquad \text{Equation1}$$

where c_i are not all zero.Multiplication of both sides of equation 1 by A yields

$$Ax_{m+1} = Ac_1x_1 + Ac_2x_2 + \ldots\ldots + Ac_mx_m$$

$$\lambda_{m+1}x_{m+1} = c_1\lambda_1x_1 + c_2\lambda_2x_2 + \ldots\ldots + c_m\lambda_mx_m \qquad \text{Equation2}$$

whereas multiplication of equation 1 by λ_{m+1} yields

$$\lambda_{m+1}x_{m+1} = c_1\lambda_{m+1}x_1 + c_2\lambda_{m+1}x_2 + \ldots\ldots + c_m\lambda_{m+1}x_m \qquad \text{Equation3}$$

Now subtracting equation 2 from equation 3 produces

$$c_1(\lambda_{m+1} - \lambda_1)x_1 + c_2(\lambda_{m+1} - \lambda_2)x_2 + \ldots\ldots\ldots\ldots + c_m(\lambda_{m+1} - \lambda_m)x_m = 0$$

and, using the fact that the first m eigenvectors are linearly independent, we conclude that all coefficients of this equation must be zero .that is,

$$c_1(\lambda_{m+1} - \lambda_1) = c_2(\lambda_{m+1} - \lambda_2) = \ldots\ldots = c_m(\lambda_{m+1} - \lambda_m) = 0$$

since all the eigenvalues are distinct, it follows that $c_i = 0, i = 1, 2, \ldots\ldots\ldots, m$.but this result contradicts our assumption

that x_{m+1} can be written as alinear combination of the first m eigenvectors.hence the set of eigenvectors

is linearly independent,and from theorem7.5 we conclude that A is diagonalizable.

Example *(10):*

Determine whether the matrix $A = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & -3 \end{pmatrix}$ is diagonalizable

Solution *:*

Since A is an upper triangular matrix,its eigenvalues are the entries of the main diagonal

$\lambda_1 = 1,\ \lambda_2 = 0,\ \lambda_3 = -3$

since these three eigenvalues are distinct by thearem above A is diagonalazable.

Remark *:*

Remember that the condition in theorem(4.4.4) is sufficient but not necessary for diagonalization.

A diagonalization matrix need not have distinct eigenvalues.

Theorem *(4.4.5):*

let A be an $n \times n$ diagonalizable matrix and P an invertible $n \times n$ matrix such that $P^{-1}AP = B$ is the diagonal form of A,

then we have

a) $B^k = P^{-1}A^kP = (P^{-1}AP)(P^{-1}AP)\ldots\ldots\ldots\ldots(P^{-1}AP)$

b) $A^k = PB^kP^{-1} = (PBP^{-1})(PBP^{-1})\ldots\ldots\ldots(PBP^{-1})$

$$(P^{-1}AP)^k = P^{-1}A^kP$$

since $P^{-1}AP = D \quad \Rightarrow \quad A = PDP^{-1}$

$$A^k = PD^kP^{-1} \qquad *$$

this last equation expresses the k^{th} power of A in terms of the k^{th} power of the diagonal matrix D.

But D^k is easy to compute for

if

$$D = \begin{pmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_n \end{pmatrix}, \text{then } D^k = \begin{pmatrix} d_1^k & 0 & 0 \\ 0 & d_2^k & 0 \\ 0 & 0 & d_n^k \end{pmatrix}$$

Example *(11);*

Use * to find A^{13} where $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

Solution *:*

we showed in before that the matrix A is diagonalizable by

$$P = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$D = P^{-1}AP = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A^{13} = PD^{13}P^{-1} = \begin{pmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2^{13} & 0 & 0 \\ 0 & 2^{13} & 0 \\ 0 & 0 & 1^{13} \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -8190 & 0 & -163 \\ 8191 & 8192 & 1638 \\ 8191 & 0 & 1638 \end{pmatrix}$$

PROBLEM SET VIII

1) Verify that A is diagonalizable by computing $P^{-1}AP$

i) $A = \begin{bmatrix} -11 & 36 \\ -3 & 10 \end{bmatrix}$, $P = \begin{bmatrix} -3 & -4 \\ -1 & -1 \end{bmatrix}$

ii) $A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 0 \\ 4 & -2 & 5 \end{bmatrix}$, $P = \begin{bmatrix} 0 & 1 & -3 \\ 0 & 4 & 0 \\ 1 & 2 & 2 \end{bmatrix}$

2) Show that the given matrix is not diagonalizable

i) $\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$

ii) $\begin{bmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 1 \\ -2 & 0 & 2 & -2 \\ 0 & 2 & 0 & 2 \end{bmatrix}$

3)Find the eigenvalues of the matrix and determine whether there are asufficient number to guarantee that the matrix is diagonalizable

$$\begin{bmatrix} 3 & 2 & -3 \\ -3 & -4 & 9 \\ -1 & -2 & 5 \end{bmatrix}$$

4)For each matrix A find (if possible) a nonsingular matrix P such that $P^{-1}AP$ is diagonal.Verify that $P^{-1}AP$
is a diagonal matrix with the eigenvalues on the diagonal.

i) $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

ii) $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}$

iii) $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 3 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

5) Find the indicated power of A

$A = \begin{bmatrix} 10 & 18 \\ -6 & -11 \end{bmatrix}$, A^6

6) Prove that if A is diagonalizable, then A^t is diagonalizable .

7) Prove that if A isdiagonalizable with n real eigenvalues $\lambda_1, \lambda_2, \ldots\ldots$ and λ_n then

$$|A| = \lambda_1, \lambda_2, \ldots\ldots \lambda_n$$

8) Find A^{11} where $A = \begin{pmatrix} -1 & 7 & -1 \\ 0 & 1 & 0 \\ 0 & 15 & -2 \end{pmatrix}$

4.5: Symmetric matrices and orthogonal Diagonoalization

Definition *4.5.1*

A square matrix A is symmetric if $A = A^t$

1) A nonsymmetric matrix may not be diagonalizable .
2) A nonsymmetric matrix can have eigenvalues that are not real
3) For a nonsymmetric matrix ,the number of L.I.N eigenvector corresponding to an eigenvalue can be less than multiplicity of the eigenvalue

Theorem *4.5.2*

If A is an $n \times n$ symmetric matrix, then the following are true:

1) A is diagonalizable.
2) All eigenvalues of A are real.
3) If λ is an eigenvalue of A with multiplicity k,then λ has k linearly independent eigenvectors. that is ,the eigenspace of λ has dimension k

Theorem(4.5.2)is called the real spectral theorem,and the set of eigenvalues of A is called the spectrum of A

Example *(12)*

Find the eigenvalues of the symmetric matrix

$$A=\begin{pmatrix} 3 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{pmatrix}$$

Solution *:*

$|\lambda I - A| = (\lambda - 4)^2(\lambda - 1)^2(\lambda - 2)$

$\Rightarrow \lambda = 4, 1, 2$

where λ_1, λ_2 are repeated twice so the eigenspace corresponding to λ_1, λ_2 are 2-diminsional and for λ_3 it 1-diminsional.

Definition *4.5.3:*

A square matrix P is called orthogonal if ito invertible and

$$P^{-1} = P^t$$

Theorem *4.5.4:*

An $n \times n$ matrix P is orthogonal *iff* its column vector form an orthonormal set.

Example *(13):*

Show that $P = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt[2]{5}} & \frac{1}{\sqrt[2]{5}} & 0 \\ \frac{-2}{3\sqrt[2]{5}} & \frac{-4}{3\sqrt[2]{5}} & \frac{5}{3\sqrt[2]{5}} \end{pmatrix}$ is orthogonal by showing that $PP^t = I$.

Then show that the column vectors of P form an orthonormal set..

Solution :

$$PP^t = \begin{pmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{-2}{3\sqrt{5}} & \frac{-4}{3\sqrt[2]{5}} & \frac{5}{3\sqrt[2]{5}} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{-2}{\sqrt{5}} & \frac{-2}{3\sqrt{5}} \\ \frac{2}{3} & \frac{1}{\sqrt{5}} & \frac{-4}{3\sqrt{5}} \\ \frac{2}{3} & 0 & \frac{5}{3\sqrt{5}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

i.e, $P^t = P^{-1}$ so P is orthogoanal

letting

$$P_1 = \left(\frac{1}{3}, \frac{-2}{\sqrt{5}}, \frac{-2}{3\sqrt{5}}\right)$$

$$P_2 = \left(\frac{2}{3}, \frac{1}{\sqrt{5}}, \frac{-4}{3\sqrt{5}}\right)$$

$$P_3 = \left(\frac{2}{3}, 0, \frac{5}{3\sqrt{5}}\right)$$

we have $\langle P_1, P_2 \rangle = \langle P_1, P_2 \rangle = \langle P_2, P_3 \rangle$

$$\|P_1\| = \|P_2\| = \|P_3\| = 1$$

Therefore $\{P_1, P_2, P_3\}$ is an orthonormal set.

Theorem *4.5.5:*

let A be an $n \times n$ symmetric matrix .If λ_1 and λ_2 are distinct eigenvalues of A ,then their corresponding eigenvectors x_1 and x_2 are orthogonal.

Proof :

let λ_1 and λ_2 be distinct eigenvalues of A with corresponding eigenvectors x_1 and x_2, Thus

$Ax_1 = \lambda_1 x_1$ and $Ax_2 = \lambda_2 x_2$.

To prove the theorem ,ito useful to start with the matrix from the dot product.

$$x_1 \bullet x_2 = \begin{bmatrix} x_{11} & x_{12} & \ldots\ldots & x_{1n} \end{bmatrix} \begin{bmatrix} x_{21} \\ x_{22} \\ \vdots \\ x_{2n} \end{bmatrix} = x_1^t x_2$$

Now we can write

$$\begin{aligned} \lambda_1(x_1.x_2) &= (\lambda_1 x_1) \bullet x_2 \\ &= (Ax_1) \bullet x_2 \\ &= (Ax_1)^t x_2 \\ &= (x_1^t A^t) x_2 \\ &= (x_1^t)(Ax_2) \\ &= x_1^t(\lambda_2 x_2) \\ &= x_1 \bullet (\lambda_2 x_2) \\ &= \lambda_2(x_1 \bullet x_2) \end{aligned}$$

This implies that $(\lambda_1 - \lambda_2)(x_1.x_2) = 0$ and since $\lambda_1 \neq \lambda_2 \quad \Rightarrow x_1 \bullet x_2 = 0$

Therefore x_1 and x_2 are orthogonal.

Example *(14):*

Show that any tow eigenvectors of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ corresponding to distinet eigenvalues are orthogonal

Solution *:*

A is syemmetric

The char. poly. of A is $|\lambda I - A| = \begin{vmatrix} \lambda - 3 & -1 \\ -1 & \lambda - 3 \end{vmatrix} = (\lambda - 2)(\lambda - 4) = 0$

$\lambda_1 = 2, \lambda_2 = 4$

Eigenvector corresponding to $\lambda = 2$ is $x_1 = s\begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad , s \neq 0$

Eigenvector corresponding to $\lambda = 4$ is $x_2 = t\begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad , t \neq 0$

Therfore $\langle x_{1,} x_2 \rangle = 1 - 1 = 0$, so x_1, x_2 are orthogonal.

Definition *4.5.6:*

A matrix A is orthogonally diagonalizable if there exists an orthogonal matrix P such that

$P^{-1}AP = D$ is diagonal .

Theorem *4.5.7:*

If A is an $n \times n$ matrix ,then the following are equivalent.
(a) A is orthogonally diagonalizable .
(b) A has an orthonormal set of n eigenvectors.
(c) A is symmetric.

Proof :
(a)⇒(b): Since A is orthogonally diagonalizable,there is an orthogonal matrix P such that $P^{-1}AP = D$ is diagonal.
As shown in the proof of Theorem 7.2.1,the n column vectrs of P are eigenvectors of A.
Since P is orthogonal, these column vectors are orthonormal (see Theorem 6.5.1),
so that A has n orthonormal eigenvectors.

(b)⇒(a): Assume that A has an orthonormal set of n eigenvectors $\{P_{1,}P_{2,}P_{3,..........,}P_n\}$.
As shown in the proof of Theorem 7.2.1,the matrix P with these eigenvectors as columns diagonally diagonalizes A.
Since these eigenvectors are orthonormal, P is orthogonal and thus orthogonally diagonalizes A.

(a)⇒(c):In the proof that (a)⇒(b) we showed that an orthogonally diagonalizable $n \times n$ matrix A is orthogonally diagonalized
by an $n \times n$ matrix P whose columns form an orthonormal set of eigenvectors of A.let D be the diagonal marix
$D = P^{-1}AP$
Thus,

$$A = PDP^{-1}$$

or,since P is orthogonal

$$A = PDP^T$$

Therefore,

$$A^T = (PDP^T)^T = PD^TP^T = PDP^T = A$$

which shows that A is symmetrice.
(c)⇒(a):The proof of this part is beyond the scope of this text and will be omitted.

Note:
If A is orthogonal
–The rowvector of A form an orthonormal set in R^n with the Eucldean inner product.
–The column vector of A from an orthonormal set in R^m with Eucldean inner product.

Example *(15): Determining whether amatrix is orthogonally diagonalizable*
Which of the following are orthogonally diagonalizable?

$$A_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 5 & 2 & 1 \\ 2 & 1 & 8 \\ -1 & 8 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 3 & 2 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad A_4 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

By theorem 7.10 the only orthogonally diagonalizable matrices are the symmetric ones : A_1 and A_4

Orthogonal Diagonalization of a Symmetric Matrix:

.Let A be an $n \times n$ symmetric matrix

1-Find all eigenvalues of A and determine the multiplicity of each.

2-For each eigenvalue of multiplicity 1,choose aunit eigenvector.(Choose any eigenvector and then normalize it.)

3-For each eigenvalue of multiplicity $k \geq 2$, find a set of k linearly independent eigenvectors.

(We know from theorem 7.7 that this is possible.)If this set is not orthonormal ,apply the Gram-Schmidt orthonormalization process.

4-The composite of steps 2 and 3 produces an orthonormal set of n eigenvectors.

Use these eigenvectors to form the columns of P.The matrix $P^{-1}AP = P^tAP = D$ will be diagonal .

(The main diagonal entries of D are the eigenvalues of A.)

Example *(16):*

Find an orthogonal matrix P that diagonalizes A,

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

Solution *:*

The char eq of A is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 4 & -2 & -2 \\ -2 & \lambda - 4 & -2 \\ -2 & -2 & \lambda - 4 \end{vmatrix} = (\lambda - 2)^2(\lambda - 8) = 0$$

so $\lambda_1 = 2, \quad \lambda_2 = 8$

$$u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

form abasis for the eigenpoe correrponding to $\lambda = 2$

Applying G.S.O.P.we get

$w_1 = u_1 = (-1, 1, 0)$

$$w_2 = u_2 - \frac{\langle u_2 \ , w_1 \rangle}{\langle w_1, w_1 \rangle} w_1$$

$$= (-1, 0, 1) - \tfrac{1}{2}(-1, 1, 0)$$

$$= (-1,0,1) + (\tfrac{1}{2},-\tfrac{1}{2},0)$$

$$= (-\tfrac{1}{2},-\tfrac{1}{2},1)$$

$$\| w_1 \| = \sqrt{1+1} = \sqrt{2}$$

$$\| w_2 \| = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}+1} = \sqrt{\tfrac{1+1+4}{4}} = \sqrt{\tfrac{6}{4}} = \sqrt{\tfrac{3}{2}}$$

$$P_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \quad , \quad P_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix}$$

the eigenspace corresponding to $\lambda = 8$ has

$u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as abasis Applaying G.S.O.P

$P_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$, so

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

which orthogonaliy diagonalizes A

$$P^T A P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

4.6 Applications of Eigenvalues and Eigenvectors systems of Differential Equations

One of the simplest differential equations is

$$y' = ay \tag{1}$$

where $y = f(x)$ is an unknown function to be determined $y' = \frac{dy}{dx}$ is its drivative and a is aconstant

(1) has infinitly many solutions ,they are the functions of the form

$$y = Ce^{ax}$$

C :constant

Each function of this form is asolution of $y' = ay$

Since $y' = Cae^{ax} = ay$

sometimes the physical problem that generates a differential equation imposes some added condition that enable us to isolate one particular solution from the general solution.

A condition which specifies the value of the solution at a point is called (an initial condition) and the problem of solving a differential equation subject to an initial condition is called (an initial-value problem).

We will concern with solving system of differential equation having the form:

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \dots\dots\dots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \dots\dots\dots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \dots\dots\dots + a_{nn}y_n \end{aligned}$$

where

$y_1 = f_1(x)$, $y_2 = f_2(x)$, ..., $y_n = f_n(x)$ are functions to be determined and the a_{ij}'s are constant.

it can be written in the form :

$$\begin{pmatrix} y_1' \\ y_2' \\ . \\ y_n' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & . & a_{1n} \\ a_{21} & a_{22} & . & a_{2n} \\ . & . & . & . \\ a_{n1} & a_{n2} & . & a_{nn} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ . \\ y_n \end{pmatrix}$$

or more briefly

$$y' = Ay$$

Example *(17):*

a) Write the following system in matrix form :

$$y_1' = 3y_1$$
$$y_2' = -2y_2$$
$$y_3' = 5y_3$$

b) Solve the system.
c) Find a solution of the system that satisfies the initial conditions:
$y_1(0) = 1 \quad , \quad y_2(0) = 4 \quad , \quad y_3(0) = -2$

Solution :

a)

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$y' = Ay$$

b)

$$y_1 = C_1 e^{3x}$$
$$y_2 = C_2 e^{-2x}$$
$$y_3 = C_3 e^{5x}$$

c)

$$1 = C_1 e^0 = C_1$$
$$4 = C_2 e^0 = C_2$$
$$-2 = C_3 e^0 = C_3$$

$$\therefore \; y_1 = e^{3x} \; , \; y_2 = 4e^{-2x} \; , \; y_3 = -2e^{5x}.$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} e^{3x} \\ 4e^{-2x} \\ -2e^{5x} \end{bmatrix}$$

If A is not diagonal we make a substitution for y that will yield to a new system with a diagonal coefficient matrix solve the new simpler system, and then use this solution to determine the solution of the original system.

4.6.1 Procedure for solving a system of First-order linear differential equation:

1) Find a matrix P that diagonallizes A.
2) Make a substituation $y = PU$ and $y' = PU'$ to obtain a new diagonal system $U' = DU$,where $D = P^{-1}AP$.
3) Solve $U' = DU$.
4) Determine y from the equation $y = PU$.

Example *(18):*

a) Solve the system:

$$y_1' = y_1 + y_2$$
$$y_2' = 4y_1 - 2y_2$$

b) Find the solution that satisfies the initial condition

$$y_1(0) = 1,\ y_2(0) = 6$$

Solution *:*

a) The coefficient matrix for the system is $A = \begin{pmatrix} 1 & 1 \\ 4 & -2 \end{pmatrix}$

A will be diagonalized by a matrix P whose columes are linearly independent eigenvectors of A

so, $|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -1 \\ -4 & \lambda + 2 \end{vmatrix} = \lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0$

$\Rightarrow \lambda = -3, 2$

the eigenvectors corresponding to $\lambda = 2$ is $p_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

if $\lambda = -3$ $\quad p_2 = \begin{pmatrix} -\frac{1}{4} \\ 1 \end{pmatrix}$

p_1, p_2 are linearly independent since $a_1 - \frac{1}{4}a_2 = 0$

$$a_1 + a_2 = 0$$

....

$$-\frac{5}{4}a_2 = 0 \quad \Rightarrow \quad a_2 = 0$$

then $a_1 = 0$

$p = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 1 \end{pmatrix}$ diagonalizes A and $D = p^{-1}Ap = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$

$$y = pU \quad \text{and} \quad y' = pU'$$

yields the new diagonal system

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = U' = DU = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix} U$$

$u_1' = 2u_1$
$u_2' = -3u_2$

from the solution of this system is :

$u_1 = C_1 e^{2x}$

$u_2 = C_2 e^{-3x}$ $\qquad U = \begin{pmatrix} C_1 e^{2x} \\ C_2 e^{-3x} \end{pmatrix}$

So, $y = PU$ yields as the solution for y

$$y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{4} \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 e^{2x} \\ C_2 e^{-3x} \end{pmatrix} = \begin{pmatrix} C_1 e^{2x} - \frac{1}{4} C_2 e^{-3x} \\ C_1 e^{2x} + C_2 e^{-3x} \end{pmatrix}$$

$y_1 = C_1 e^{2x} - \frac{1}{4} C_2 e^{-3x}$
$y_2 = C_1 e^{2x} + C_2 e^{-3x}$

b) $\quad y_1(0) = 1 \quad , \quad y_2(0) = 6$

$1 = C_1 - \frac{1}{4} C_2$
$6 = C_1 + C_2$
$\Rightarrow C_1 = 2 \quad , \quad C_2 = 4$

then $y = 2e^{2x} + 4e^{-3x}$

Example *(19):*

Solve the following system of linear differential equations :-

$y_1' = 3y_1 + 2y_2$
$y_2' = 6y_1 - y_2$

Solution *:*

First we find a matrix P that diagonalizes $\quad A = \begin{bmatrix} 3 & 2 \\ 6 & -1 \end{bmatrix}$

The eigenvalues of A are $\lambda_1 = -3$,and $\lambda_2 = 5$, with corresponding eigenvectors $v_1 = (1,-3)$ and $v_2 = (1,1)$. Since the eigenvalues are distinct.
we know that we can diagonalize A by using the matrix p whose columns consist of the eigenvectors v_1 and v_2. That is,

$$P = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \quad \text{and} \quad D = P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix}$$

The system represented by $w' = P^{-1}APw$ has the following form

$$\begin{bmatrix} w_1' \\ w_2' \end{bmatrix} = \begin{bmatrix} -3 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad \Rightarrow \quad w_1' = -3w_1,\ w_2' = 5w_2$$

The solution to this system of equation is

$$w_1' = C_1 e^{-3t}$$
$$w_2' = C_2 e^{5t}$$

To return to the original variable y_1 and y_2 , we use the substitution $y = pw$ and write

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

which implies that the solution is

$$y_1 = \quad w_1 + w_2 = \quad C_1 e^{-3t} + C_2 e^{5t}$$
$$y_2 = -3w_1 + w_2 = -3C_1 e^{-3t} + C_2 e^{5t}.$$

4.6.2 Quadratic Form:

Aquadratic form in tow variable x and y is defined to be

$$ax^2 + 2bxy + cy^2 \qquad 1$$

The following are quadratic forms in x and y

$$2x^2 + 6xy - 7y^2 \quad a = 2,\ b = 6,\ c = -7$$
$$4x^2 - 5y^2 \quad a = 4,\ b = 0,\ c = -5$$
$$xy \quad a = 0,\ b = \tfrac{1}{2},\ c = 0$$

(1) can be written in the form

$$ax^2 + 2bxy + cy^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^T A x$$

the digonal entries the coefficients of the squared terms and the entries off the main digonal are each half the coefficient of the product term xy

$$2x^2 + 6xy - 7y^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$4x^2 - 5y^2 = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$xy = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Definition *4.6.3:*

Aquadretic form in the n variables $x_1, x_2, \ldots\ldots\ldots, x_n$ is an expression that can be written as

$$[x_1 x_2 \ldots\ldots\ldots\ldots x_n] A \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x^T A x \tag{2}$$

where A is asymmutric $n \times n$ matrix.

(2) can be written more compactily as $x^T A x$

$$x^T A x = a_{11} x_1^2 + a_{22} x_2^2 + \ldots + a_{nn} x_n^2 + \sum a_{ij} x_i x_j$$

where $\sum_{i \neq j} a_{ij} x_i x_j$ denotes the sum of all terms of the form $a_{ij} x_i x_j$ where $x_i, x_j$are different variables.

The term $a_{ij} x_i x_j$ are called the cross product terms of the quadratic form.

Symmetric matrices are useful but not essential for reoresent quadratic forms

for example $2x^2 + 6xy - 7y^2$ we might split the coefficint of the cross product term into $5 + 1$ or $4 + 2$ then

$$2x^2 + 6xy - 7y^2 = [xy] \begin{bmatrix} 2 & 5 \\ 1 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

or

$$2x^2 + 6xy - 7y^2 = [xy] \begin{bmatrix} 2 & 4 \\ 2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

we will always use symmetric matrices, when we denote $x^T A x$ i.e, A is symmetric

Remark *4.6.4:*

If A is symmetric then $A = A^T$, then $x^T A x = x^T (Ax) = \langle Ax, x \rangle = \langle x, Ax \rangle$

Example *(20):*

The following is a quadratic form in x_1, x_2 and x_3 varible

$$x_1^2 + 7x_2^2 - 3x_3^2 + 4x_1 x_2 - 2x_1 x_3 + 6x_2 x_3 = [x_1 x_2 x_3] \begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & 3 \\ -1 & 3 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Note that the coefficient of the sequared terms appear on the main diagonal of the3×3 matrix

$$\begin{bmatrix} \text{coefficient of} & \text{position in matrix } A \\ x_1x_2 & a_{12} \text{ and } a_{21} \\ x_1x_3 & a_{13} \text{ and } a_{31} \\ x_2x_3 & a_{23} \text{ and } a_{32} \end{bmatrix}$$

4.6.5 Problems involving quadratic forms:

Theorem *3.6.6:*

let A be a symmetric $n \times n$ matrix whose eigenvalues in decreasing size order are $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$.

if x is constrainted so that $\|x\| = 1$ relative to the Eucliden inner product on R^n then:

a) $\lambda_1 \geq x^T Ax \geq \lambda_n$

b) $x^T Ax = \lambda_n$ if x is an eigenvector of A corresponding to λ_n and $x^T Ax = \lambda_1$ if x is an eigenvector of A corresponding to λ_1.

it follows from that therorem that subject to the constraint

$\|x\| = (x_1^2 + x_2^2 + \ldots + x_n^2)^{\frac{1}{2}} = 1$

the quadratic form $x^T Ax$ has a maximum value of λ_1(the largest eigenvalue) and a minimum value of λ_n (the smallest eigenvalue).

Example *(21):*

Find the maximum and minumum values of the quadratic form $x_1^2 + x_2^2 + \ldots + 4x_1x_2$ subject to the constraint

$x_1^2 + x_2^2 = 1$, and the values of x_1 and x_2 at which the maximum and minimum occurs

$$x_1^2 + x_2^2 + 4x_1x_2 = [x_1x_2]\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution *:*

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & -2 \\ -2 & \lambda - 1 \end{vmatrix} = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$$

then the eigenvalues of A are $\lambda = 3, \lambda = -1$ which are the maximum and the miniumum values respectivily of the quadratic

form subject to the contraint. To find values of x_1 and x_2 we find the eigenvalues corresponding to eigenvalues

then normalize then to satisfy the condition

$x_1^2 + x_2^2 = 1$

if $\lambda = 3$ $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, if $\lambda = -1$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

normolizing these eigenvalues

$$\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{bmatrix}$$

Thus subject to constraint $x_1^2 + x_2^2 = 1$
The maximum value of the quadratic form is $\lambda = 3$ which occurs if $x_1 = \frac{1}{\sqrt{2}}$, $x_2 = \frac{1}{\sqrt{2}}$ and the minumum value
is $\lambda = -1$ which occurs if $x_1 = \frac{1}{\sqrt{2}}$, $x_2 = -\frac{1}{\sqrt{2}}$

Definition *4.6.7:*

A quadratic form $x^T Ax$ is called positive definite if $x^T Ax \succ 0$ for all $x \neq 0$, and a symmetric matrix A is called
a positive definite matrix if $x^T Ax$ is a positive definite quadratic form.

Theorem *6.6.8:*

A symetric matrix A is positive difinite *iff* all the eigenvalues of A are positive.

Example *(22):*

In ex. before we showed that $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$

has eigiivalues $\lambda = 2, 8$ since these are positive, the matrix A is definite and for all $x \neq 0$
$x^T Ax = 4x_1^2 + 4x_2^2 + 4x_3^2 + 4x_1x_2 + 4x_1x_3 + 4x_2x_3 > 0$

if $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}$ is a square matrix

then the preinicible submatrices of A are the submatrices from the first r rows and r colums of A for $r = 1, 2, , \dots, n$
These submatrices are

$$A_1 = [a_{11}] , A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} , A_3 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\ldots\ldots\ldots A_n = A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & : & : \\ a_{n1} & a_{n2} & .. & a_{nn} \end{bmatrix}$$

Theorem *4.6.9:*

A symmetric matrix A is positive definitie *iff* the detrminant of every principal submatrix is positive.

Example *(23):*

The matrix $A = \begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{bmatrix}$

is positive definite since

$|A_1| = |2| = 2$, $|A_2| = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$

, $|A_3| = \begin{vmatrix} 2 & -1 & -3 \\ -1 & 2 & 4 \\ -3 & 4 & 9 \end{vmatrix} = 1$

all of which are positive, thus all eigavalues of A are positive and $x^T Ax > 0$ for all $x \neq 0$

$2x_1^2 + 2x_2^2 + 9x_3^2 - 2x_1x_2 - 6x_1x_3 + 8x_2x_3 > 0$

4.7:Diagonalizing Quadratic forms ,Conic sections:

Lit $x^T Ax$ be a quadratic form in the variables $x_1, x_2, \ldots\ldots. x$ where A is symmetric : if Porthogonally diagonalizes A
and if the new variabiles $y_1, y_2, \ldots\ldots, y_n$ are defined by the equation $x = py$,
then substituting this equation in $x^T Ax$ yields

$$x^T Ax = y^T Dy = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots\ldots + \lambda_n y_n^2$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ are the eigenvalues of A and

$$D = p^T Ap = \begin{pmatrix} \lambda_1 & 0 & \ldots\ldots & 0 \\ 0 & \lambda_2 & \ldots\ldots & 0 \\ \vdots & \vdots & . & \vdots \\ 0 & 0 & \ldots\ldots & \lambda_n \end{pmatrix}$$

the matrix P orthogonally diagonalizes the quadratic form or reduse the quadratic form to a sum of squares.

Example *.(24):*

Find a change of variables that will reduce the quadratic form $x_1^2 - x_3^2 - 4x_1x_2 + 4x_2x_3$ to the sum of squares
and express the quadratic form in terms of the new variables?

Solution *:*

the quadratic form is written as $\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ -2 & 0 & 2 \\ 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

$$|\lambda I - A| = \begin{vmatrix} \lambda - 1 & 2 & 0 \\ 2 & \lambda & -2 \\ 0 & -2 & \lambda + 1 \end{vmatrix} = \lambda^3 - 9\lambda = \lambda(\lambda^2 - 9) = \lambda(\lambda - 3)(\lambda + 3) = 0$$

$\lambda = 0 \quad ,\lambda = -3 \quad ,\lambda = 3$

$$v_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \Rightarrow p_1 = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v_2 = \begin{pmatrix} -1 \\ -2 \\ 2 \end{pmatrix} \Rightarrow p_2 = \begin{pmatrix} \frac{-1}{3} \\ \frac{-2}{3} \\ \frac{2}{3} \end{pmatrix}$$

$$v_3 = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \Rightarrow p_3 = \begin{pmatrix} \frac{-2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}$$

$$\therefore\ P = \begin{pmatrix} \frac{2}{3} & \frac{-1}{3} & \frac{-2}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

$$X = PY$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = P\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$$

$$D = p^T A p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$\therefore\ y^T D y = x^T A x = -3y_2^2 + 3y_3^2$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} D \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

4.7.2:Eliminating the cross product term

Every conic section in the xy plane has an equation of the form:

$$ax^2 + cy^2 + 2bxy + dx + ey + f = 0$$

can be written in the form

$$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} d & e \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + f = 0$$

or

$$x^T A x + k x + f = 0$$

where

$$x = \begin{bmatrix} x \\ y \end{bmatrix}, \; A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \; k = \begin{bmatrix} d & e \end{bmatrix}$$

Now consider aconic C whose eq. in xy-coordinate is

$$x^T A x + k x + f = 0$$

we would like to rotate the xy-coordinate axes so that the eq. of the conic in the new $x' y'$-coordinate
system has no cross product term this can be done follows:

step(1):

Find a matrix $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ that orthogonally diagonalizes the quadratic form $x^T A x$

step(2):

Interchange the columns of P ,if necessary ,to make $det(p) = 1$.This assures that the orthogonal coordinate transformation ,$|p| = 1$

$$x = Px', \quad \text{that is} \quad , \begin{bmatrix} x \\ y \end{bmatrix} = P \begin{bmatrix} x' \\ y' \end{bmatrix} \tag{5}$$

is a rotation.

step(3):

To obtain the equation for C in the $x' y'$-system ,substitute (5) into (4) this yields

$$(Px')^T A (Px') + k(Px') + f = 0$$

or

$$(x')^T (p^T A p) x' + f = 0 \tag{6}$$

since P orthogonally diagonalizes A,

$$P^T A P = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

where $\lambda_1$and λ_2 are eigenvalues of A.thus ,(6) can be rewritten as

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} d' & e' \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + f = 0$$

or

$$\lambda_1 x'^2 + \lambda_2 y'^2 + d' x' + e' y' + f = 0$$

(where $d' = dp_{11} + ep_{21}$ and $e' = dp_{12} + ep_{22}$). This equation has no cross-poduct term. The following theorm summarizes this discussion.

Theorem *4.7.3 (Principle Axes Theorem for R^2)*

Let

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0$$

be the equation of a conic C,and let

$$x^T A x = ax^2 + 2bxy + cy^2$$

be the associated quadratic form .then the coordinate axes can be rotated so that the equation for C
in the new $x' y'$-coordinate system has the form

$$\lambda_1 x'^2 + \lambda_2 y'^2 + d' x' + e' y' + f = 0$$

where $\lambda_1$and$\lambda_2$are the eigenvalues of A.The rotation can be accomplished by the substitution

$$x = Px'$$

where p orthogonally diagonalizes $x^T A x$ and $det(P) = 1$.

Example *(25) :*

Describe the conic C whose equation is $5x^2 - 4xy + 8y^2 - 36 = 0$ " This equation in the form $x^T A x - 36 = 0$ "

$$A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix}$$

Solution *:*

$$|\lambda I - A| = \begin{vmatrix} \lambda - 5 & 2 \\ 2 & \lambda - 8 \end{vmatrix} = (\lambda - 9)(\lambda - 4) = 0$$

if $\lambda = 4$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$$

$$x_1 - 2x_2 = 0$$
$$x_1 = 2x_2 = 2t$$
$$x_2 = t$$
$$x = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

if $\lambda = 9$

$$x = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad , \quad v = \begin{pmatrix} \frac{-1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \Rightarrow |P| = 1$$

$$P^T A P = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}$$

$$\begin{bmatrix} x' & y' \end{bmatrix}\begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} - 36 = 0$$

$$4x'^2 + 9y'^2 - 36 = 0$$
$$4x'^2 + 9y'^2 = 36$$
$$\frac{x'^2}{3^2} + \frac{y'^2}{4^2} = 1$$

which is an equation of an ellipse

Example *(26):*

Describe the conic C whose equation is

$$5x^2 - 4xy + 8y^2 + \frac{20}{\sqrt{5}}x + \frac{80}{\sqrt{5}}y + 4 = 0$$

Solution *:*

The matrix form of this equation is

$$x^T A x + k x + 4 = 0$$

$$A = \begin{pmatrix} 5 & -2 \\ -2 & 8 \end{pmatrix} \quad , \quad k = \begin{bmatrix} d & e \end{bmatrix} = \begin{bmatrix} \frac{20}{\sqrt{2}} & , & \frac{-80}{\sqrt{5}} \end{bmatrix}$$

we find P first eigenvalue then the eigenvector normalize then we find

$$P = \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} \quad , \quad |P| = 1$$

$$P^T A P = \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \quad , \quad kp = \begin{bmatrix} \frac{20}{\sqrt{5}} & \frac{-80}{\sqrt{5}} \end{bmatrix} \begin{pmatrix} \frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{pmatrix} = \begin{pmatrix} -8 & -36 \end{pmatrix}$$

$$\begin{bmatrix} x' & y' \end{bmatrix} \begin{pmatrix} 4 & 0 \\ 0 & 9 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{bmatrix} -8 & -36 \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} + 4 = 0$$

$$4x'^2 + 9y'^2 - 8x - 36y + 4 = 0$$
$$4x'^2 - 8x' + 9y'^2 - 36y' = -4$$
$$4(x'^2 - 2x') + 9(y'^2 - 4y') = -4$$
$$4(x'^2 - 2x' + 1) + 9(y'^2 - 4y' + 4) = -4 + 4 + 36$$
$$4(x' - 1)^2 + 9(y' - 2)^2 = 36$$

we translate the cordinate axes by many of the translation equations
let $x' - 1 = x''$, $y' - 2 = y''$
then

$$4x''^2 + 9y''^2 = 36$$

$$\frac{x''^2}{3^2} + \frac{y''^2}{2^2} = 1$$

which is an equation of the ellipse

Note:

$\frac{x^2}{k^2} + \frac{y^2}{l^2} = 1$, $k, l > 0$ is an equation of Ellipse or circle

$\frac{x^2}{k^2} - \frac{y^2}{l^2} = 1$, $k, l > 0$ Hyperbola

$\frac{y^2}{k^2} - \frac{x^2}{l^2} = 1$, $k, l > 0$ Hyperbola

$y^2 = kx$ prapola

$x^2 = ky$ prapola

4.7.4 Quadric Surfaces

An equation of the form:

$$ax^2 + by^2 + cy^2 + 2dxy + 2exz + 2fyz + gx + hy + iz + j = 0 \qquad 1$$

where $a, b, \ldots\ldots, f$ are not all zero is called a quadratic eq.in x, y and z , the expression

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz$$

is called the associated quadretic form eq.(1) can be written in the matrix form

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + j = 0$$

or

$$x^T A x + K x + j = 0$$

where

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad A = \begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}, K = \begin{bmatrix} g & h & i \end{bmatrix}$$

Example *(27):*

The quadratic form associated with the quadratic equation

$$3x^2 + 2y^2 - z^2 + 4xy + 3xz - 8yz + 7x + 2y + 3z - 7 = 0$$

is:

$$3x^2 + 2y^2 - z^2 + 4xy + 3xz - 8yz$$

graphs of quadric equations in x, y and zero called quadratics or quadric surfaces.

Example *(28):*

Describe the quadric surface where equation is :

$$4x^2 + 36y^2 - 9z^2 - 16x - 216y + 304 = 0$$

$$4(x^2 - 9x) + 36(y^2 - 6y) - 9z^2 = -304$$

$$4(x^2 - 4x + 4) + 36(y^2 - 6y + 9) - 9z^2 = -304 + 16 + 334$$

$$4(x-2)^2 + 36(y-3)^2 - 9z^2 = 36$$

$$\frac{(x-2)^2}{9} + \frac{(y-3)^2}{1} - \frac{z^2}{4} = 1$$

Translating the axes λ by means of the transtalion eq.

$$x' = x - 2 \ , y' = y - 3 \ , \ z' = z$$

yields to:

$$\frac{x'^2}{9} + y'^2 - \frac{z'^2}{4} = 1$$

which is the eq. of a hyperboloid.

4.7.5 ELIMINATING Cross product terms:-

let Q be a quedric surface whose eq. in xyz-coordinates is

$$x^T A x + kx + j = 0 \qquad 2$$

we want to rotate the xyz - coordinate axis to
the new $x'y'z'$ - coordinate system has no cross
product terms by

1. Find a matrix P that orthogonally diagonalizes $x^T A x$

2. Interchange two columms of P, if necessary,
to make $\det(P) = 1$

$$x = Px', \text{ that is } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = P\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \qquad 3$$

is a votation

3 Substitute (3) in to (2) this will producce the eq.
for the quadric in $x'y'z'$ - coordinates with no-cross.product
terms.

Theorem *4.7.6(principal Axes theorem for R^3):*

let

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz + gx + hy + iz + j = 0$$

be the eq. of a quadric Q and let

$$x^T Ax = a^2 x + by^2 + cz^2 + 2dxy + 2exz + 2fyz$$

be the associated quadratec form.The coordinate axis can be rotated so that the eq. of Q in the $x' y' z'$ coordinate system has the form

$$\lambda_1 x_1'^2 + \lambda_2 y'^2 + \lambda_3 z'^2 + g' x' + h' y' + i' z' + j = 0$$

where $\lambda_1, \lambda_2, \lambda_3$ are eigenvalues of A .The rotation can be accomplished by the substitution

$$x = Px'$$

where P orthogonally diagonalizes $x^T Ax$ and $\det(P) = 1$

Example *(29):*

Describe the quadric surface whose equation is

$$4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz - 3 = 0$$

The matrix form is

$$x^T A x - 3 = 0$$

where

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

the eigenvalues of A are $\lambda = 2, 8$ and A is orthogonally

diagonialzable by $P = \begin{pmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix}$, $|P| = 1$ verify??

$$x = Px'$$

$$(Px')^T A (Px') - 3 = 0$$

$$x'^T (P^T A P) x' - 3 = 0 \qquad 5$$

$$P^T A P = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

so (5) becomes

$$\begin{bmatrix} x' & y' & z' \end{bmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} - 3 = 0$$

$$2x'^2 + 2y'^2 + 8z'^2 = 3 \Rightarrow \frac{x'^2}{3/2} + \frac{y'^2}{3/2} + \frac{z'^2}{3/8} = 1$$

which is eq. of an ellipsoid.

PROBLEM SET IX

(1) Determine whether the given matrix is symmetric.

(i) $\begin{bmatrix} 1 & -1 \\ -1 & 4 \end{bmatrix}$

(ii) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -3 \\ 2 & -3 & 0 \end{bmatrix}$

(2) Find the eigenvalues of the given symmetric matrix. For each eigenvalue , find the dimension of the cooresponding eigenspace.

(i) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

(ii) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(3) Determine whether the given matrix is orthogonal.

(i) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$

(ii) $\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{-\sqrt{6}}{6} & \frac{\sqrt{3}}{3} \\ 0 & \frac{\sqrt{6}}{3} & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} & \frac{-\sqrt{3}}{3} \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{1}{10}\sqrt{10} & 0 & 0 & \frac{-3}{10}\sqrt{10} \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{3}{10}\sqrt{10} & 0 & 0 & \frac{1}{10}\sqrt{10} \end{bmatrix}$

(4) Find an orthogonal matrix P such that P^tAP diagonalizes A
Verify that P^tAP gives the proper diagonal form.

(i) $A = \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 & 10 & 10 \\ 10 & 5 & 0 \\ 10 & 0 & -5 \end{bmatrix}$

(5) Prove that if A is an $m \times n$ matrix,then A^tA and AA^t are symmetric.

(6) Prove that if A is an orthogonal matrix ,then $|A| = \pm 1$

(7) Solve the given system of first-order linear differential equations.

(i) $\begin{aligned} y_1' &= 2y_1 \\ y_2' &= y_2 \end{aligned}$

(ii) $\begin{aligned} y_1' &= -y_1 \\ y_2' &= 6y_2 \\ y_3' &= y_3 \end{aligned}$

(8) Solve the given system of first-order linear differential equations.

(i) $\begin{aligned} y_1' &= y_1 - 4y_2 \\ y_2' &= 2y_2 \end{aligned}$

(ii) $\begin{aligned} y_1' &= -3y_2 + 5y3 \\ y_2' &= -4y_1 + 4y_2 - 10y_3 \\ y_3' &= 4y_3 \end{aligned}$

(9) Write out the system of first-order linear differential equations represented by the matrix equation $y' = Ay$ .Then verify the indicted general solution .

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{bmatrix}, \quad \begin{aligned} y_1 &= C_1 + C_2 \cos 2t + C_3 \sin 2t \\ y_2 &= 2C_3 \cos 2t - 2C_2 \sin 2t \\ y_3 &= -4C_2 \cos 2t - 4C_3 \sin 2t \end{aligned}$$

(10) Find the matrix of the quadratic form associated with the given equation

(i) $9x^2 + 10xy - 4y^2 - 36 = 0$

(ii) $10xy - 10y^2 - 4x - 48 = 0$

(11) Find the matrix A of the quadratic form associated with the given equation. In each case eigenvalues of A and an orthogonal matrix P such that P^tAP is diagonal.

(i) $13x^2 + 6\sqrt{3}\,xy + 7y^2 - 16 = 0.$

(ii) $16x^2 - 24xy + 9y^2 - 60x - 80y + 100 = 0.$

(12) Use the Principal Axes Theorm to perform a rotation of axes to eliminate the xy−term in the given quadratic equation .Identify the resulting rotated conic and give its equation in the new coordinate system.

(i) $13x^2 - 8xy + 7y^2 - 45 = 0$

(ii) $2x^2 + 4xy + 2y^2 + 6\sqrt{2}\,x + 2\sqrt{2}\,y + 4 = 0$

(iii) $xy + x - 2y + 3 = 0$

(13) Find the matrix A of the quadratic form associated with the given equation.Then find the equation of the rotated quadric surface in which the xy, xz and yz terms have been eliminated.

(i) $3x^2 - 2xy + 3y^2 + 8z^2 - 16 = 0$

Problem Set X

(1) In each part find a change variables that reduces the quadratic form to a sum or difference of square and express the quadratic form in terms of the new variables:
(a) $5x_1^2 + 2x_2^2 + 4x_1x_2 = 0$ (b) $3x_1^2 + 4x_2^2 + 5x_3^2 + 4x_1x_2 - 4x_2x_3 = 0$
(c) $2x_1x_3 + 6x_2x_3 = 0$

(2) Find the quadratic form associated with the following then express each of the quadratic equations in the matrix form:

$$x^T Ax + kx + f = 0$$

(a) $2x^2 - 3xy + 4y^2 - 7x + 2y + 7 = 0$ (b) $y^2 + 7x - 8y - 5 = 0$

(3) Express each of the quadratic equation in the matrix form :

$$x^T Ax + kx + j$$

$$3x^2 + 7z^2 + 2xy - 3xz + 4yz - 3x = 4$$

(4) In each part determine the transtation equations that will put the quadric in standard posetion:
(a) $3x^2 - 3y^2 - z^2 + 42x + 144 = 0$
(b) $9x^2 + 36y^2 + 4z^2 - 18x - 144y - 24z + 153 = 0$

(5) In the following find a rotation $x = px'$ that removes the cross-product term and give its equations in the $x'y'z'$ system:
(a) $4x^2 + 4y^2 + 4z^2 + 4xy + 4xz + 4yz - 5 = 0$
(b) $2xy + z = 0$

Chapter X
Complex Vector Space

5.1: Complex number:

For many important application of vector its desirable to allow the scalors to be complex numbers
a vector spase that allow complex scalars is called a complex vector space.

In the begining of this chapter we will review some of the basis properties of comblex numbers.

since $x^2 \geq 0$ for every real number x, the equation $x^2 = 1$

has no real solution Todeal with this problem mathematicias of the eighteenth century introduced the imaginary number

so

$$i = \sqrt{-1}$$

$$i^2 = (\sqrt{-1})^2 = -1$$

Definition *(5.1.1):*

A complex number is an ordered pair of real numbers , denoted either by (a, b) or $a + bi$.

we denote the complex number by z,so

$$z = a + bi$$

where a is called the real part of z denoted by $\mathrm{Re}(z)$ and b is the imaginary part of z denoted by $Im(z)$.

For example:

$Re(4 - 3i) = 4$ and $Im(4 - 3i) = -3$

when complex numbers are represented geometrically in an xy-coordniate system ,
the x-axis is called the real axis , the y-axis is the imaginary axis and the plane is called the complex plane.

Definition *5.1.2:*

Two complex numbers $a + bi$ and $c + di$ are equal $a + bi = c + di$ if $a = c$ and $b = d$

propirties of complex numbers:

i) $(a + bi) + (c + di) = (a + c) + (b + d)i$

ii) $(a + bi) - (c + di) = (a - c) + (b - d)i$

iii) $k(a + bi) = (ka) + (kb)i$, k real

iv) $(-1)z + z = 0$

v) $(-1)z = -z$, is the negative of z

Example *(1):*

If $z_1 = 4 - 3i$ and $z_2 = -2 + 5i$ find $z_1 + z_2$, $z_1 - z_2$, $3z_1$, and $-z_2$

Solution *:*

$z_1 + z_2 = 2 + 2i$

$z_1 - z_2 = 6 - 8i$

$3z_1 = 12 - 9i$

$-z_2 = 2 - 5i$

Multiplication of complex number

$$(a+bi)(c+di) = ac + bdi^2 + bci + adi$$
$$= (ac - bd) + (ad + bc)i$$

Example *(2):*

Find $(3+2i)(4+5i) = (12-10) + (15+8)i$
$= 2 + 23i$

find i^2

$i^2 = (0+i)(0+i) = (0.0 - 1.1) + (0.1 + 0.1) = -1$

Verify the following:

$z_1 + z_2 = z_2 + z_1$

$z_1 z_2 = z_2\, z_1$

$z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

$z_1(z_2 z_3) = (z_1 z_2) z_3$

$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$

$0 + z = z$

$z + (-z) = 0$

$1.z = z$

Example *(3):*

If $A = \begin{pmatrix} 1 & -i \\ 1+i & 4-i \end{pmatrix}$ and $B = \begin{pmatrix} i & 1-i \\ 2-3i & 4 \end{pmatrix}$

then

$$A + B = \begin{pmatrix} 1+i & 1-2i \\ 3-2i & 8-i \end{pmatrix}$$

$$A - B = \begin{pmatrix} 1-i & -1 \\ -1+4i & -i \end{pmatrix}$$

$$iA = \begin{pmatrix} i & 1 \\ -1+i & 1+4i \end{pmatrix}$$

$$AB = \begin{pmatrix} -3-i & 1-5i \\ 4-13i & 18-i \end{pmatrix}$$

PROBLEM SET XI

(1) In each part plot the point and sketch the vector corresponds to the given coplex number.

(a) $2+3i$ (b) $-3-2i$

(2) In each part use the given information to find the real numbers x and y.

(a) $x-iy=-2+3i$ (b) $(x+y)+(x-y)i=3+i$

(3) Given that $z_1 = 1-2i$ and $z_2 = 4+si$ find

(a) z_1+z_2 (b) z_1-z_2 (c) $4z_1$

(d) $-z_2$ (e) $3z_1+4z_2$ (f) $\frac{1}{2}z_1-\frac{3}{2}z_2$

(4) In each part solve for z.

(a) $z+(1-i) = 3+2i$ (b) $-5z = 5+10i$ (c) $(i-z)+(2z-3i) = -2+7i$

(5) In each part find real numbers k_1 and k_2 that satisfy the equation.

$k_1(2+3i)+k_2(1-4i) = 7+5i$

(6) In each part find z_1z_2, z_1^2, and z_2^2

(a) $z_1 = 3i$, $z_2 = 1-i$ (b) $z_1 = \frac{1}{3}(2+4i)$, $z_2 = \frac{1}{2}(1-5i)$

(7) Perform the calculations and express the result in the form $a+bi$

(i) $(1+2i)(4-6i)^2$

(ii) $i(1+7i)-3i(4+2i)$

(iii) $(1+i+i^2+i^3)^{100}$

(8) Let

$$A = \begin{bmatrix} 1 & i \\ -i & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 2+i \\ 3-i & 4 \end{bmatrix}$$

Find

(a) $A+3iB$ (b) B^2-A^2

(9) Let

$$A = \begin{bmatrix} 3+2i & 0 \\ -i & 2 \\ 1+i & 1-i \end{bmatrix} \quad B = \begin{bmatrix} -i & 2 \\ 0 & i \end{bmatrix} \quad C = \begin{bmatrix} -1-i & 0 & -i \\ 3 & 2i & -5 \end{bmatrix}$$

Find

(a) $A(BC)$ (b)$(CA)B^2$ (C) $(1+i)(AB)+(3-4i)A$

5.2: Modulus,Complex conjugate,Division

Definition *5.2.1:*

If $z = a + bi$ is any complex number ,then the congugate of z denoted by $\bar{z}$ is defined by

$$\bar{z} = a - bi$$

if $z = 3 + 2i$, $\bar{z} = 3 - 2i$
$z = -4 - 2i$, $\bar{z} = -4 + 2i$
$z = i$, $\bar{z} = -i$
$z = 4$, $\bar{z} = 4$

Definition *5.2.2:*

The modulus of a complex number $z = a + bi$,denoted by $|z|$, is defined by

$$|z| = \sqrt{a^2 + b^2}$$

The modulus of real number is simply its absolute value

Example *:*

Find $|z|$ if $z = 3 - 4i$
$|z| = \sqrt{9 + 16} = \sqrt{25} = 5$

Theorem *5.2.3:*

For any compex number z,
$z\bar{z} = |z|^2$

Proof :

If $z = a + bi$,then
$z\bar{z} = (a + bi)(a - bi)$
$= a^2 - abi + abi - b^2 i^2$
$= a^2 + b^2 = |z|^2$

Theorem *5.2.4:*

If $z_2 \neq 0$, then equation $z_1 = z_2 z$ has a unique solution which is ($z = \frac{z_1}{z_2}$)

$$z = \frac{1}{|z_2|^2} z_1 \bar{z}_2$$

Proof :

let $z = x + iy$, $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,then
$z_1 = z_2 z$ will be

$$(x_1 + iy_1) = (x_2 + iy_2)(x + iy)$$

or

$$x_1 + iy_1 = (x_2x - y_2y) + i(y_2x + x_2y)$$

on equating real and imaginary parts

$$x_2x - y_2y = x_1$$
$$y_2x + x_2y = y_1$$

i.e

$$\begin{pmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

since $z_2 = x_2 + iy_2 \neq 0$ it follows that $x_2$and y_2 are not both zeros so

$$\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix} = x_2^2 + y_2^2 \neq 0$$

Thus by cramers rule

$$x = \frac{\begin{vmatrix} x_1 & -y_2 \\ y_1 & x_2 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} = \frac{x_1x_2 + y_1y_2}{|z_2|^2}$$

$$y = \frac{\begin{vmatrix} x_2 & x_1 \\ y_2 & y_1 \end{vmatrix}}{\begin{vmatrix} x_2 & -y_2 \\ y_2 & x_2 \end{vmatrix}} = \frac{x_2y_1 - x_1y_2}{x_2^2 + y_2^2} = \frac{x_2y_1 - x_1y_2}{|z_2|^2}$$

Thus $z = x + iy = \frac{1}{|z_2|^2}[(x_1x_2 + y_1y_2) + i(x_2y_1)]$

$= \frac{1}{|z_2|^2}(x_1 + iy_1)(x_2 - iy_2)$

$= \frac{1}{|z_2|^2} z_1\bar{z}_2$

Thus for $z_2 \neq 0$ we define

$$\frac{z_1}{z_2} = \frac{1}{|z_2|^2} z_1\bar{z}_2$$

Example *(4):*

Express $\frac{3+4i}{1-2i}$ in the form $a + bi$

Solution *:*

$\frac{3+4i}{1-2i} = \frac{1}{|1-2i|^2}(3+4i)(1+2i)$

$= \frac{1}{5}(-5+10i)$

$= -1+2i$

or

$\frac{3+4i}{1-2i} \cdot \frac{1+2i}{1+2i} = \frac{-5+10i}{5} = -1+2i$

Example *(5):*

Use Cramers Rule to solve.

$$ix + 2y = 1 - 2i$$
$$4x - iy = -1 + 3i$$

Solution *:*

$x = \frac{|A_1|}{|A|}, \quad y = \frac{|A_2|}{|A|}, \quad |A| = \begin{vmatrix} i & 2 \\ 4 & -i \end{vmatrix} = -7$

$$|A_1| = \begin{vmatrix} 1-2i & 2 \\ -1+3i & -i \end{vmatrix} = -i - 2 + 2 - 6i = -7i$$

$\therefore\ x = \frac{-7i}{-7} = i$

$$|A_2| = \begin{vmatrix} i & 1-2i \\ 4 & -1+3i \end{vmatrix} = -i - 3 - 4 + 5i = -7 + 7i$$

$\therefore\ y = \frac{-7+7i}{-7} = 1 - i$

Theorem *5.2.5:*

For any complex numbers z , z_1, and z_2

a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

b) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$

c) $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

Proof :

(a) Let $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$,then

$$\begin{aligned}\overline{(z_1 + z_2)} &= \overline{(a_1 + a_2) + (b_1 + b_2)i} \\ &= (a_1 + a_2) - (b_1 + b_2)i \\ &= (a_1 - b_1 i) + (a_2 - b_2 i) \\ &= \overline{z_1} + \overline{z_2}\end{aligned}$$

(c) $\overline{z_1}\,\overline{z_2} = \overline{(a_1a_2 - b_1b_2) + (b_1a_2 + a_1b_2)i}$

$= (a_1a_2 - b_1b_2) - (b_1a_2 + a_1b_2)i$

$$= (a_1 - b_1 i)(a_2 - b_2 i)$$
$$= \overline{z_1}\,\overline{z_2}$$

(e) $\overline{\overline{z}} = \overline{(a - bi)}$
$$= a + bi$$
$$= z$$

Remark *4.2.4:*

$$\overline{z_1 + z_2 + \ldots\ldots + z_n} = \overline{z_1} + \overline{z_2} + \ldots\ldots + \overline{z_n}$$

$$\overline{z_1 z_2 \ldots\ldots z_n} = \overline{z_1}\,\overline{z_2} \ldots\ldots \overline{z_n}$$

5.3 Polar Form;Demoivrs Theorem:

If $z = x + iy$ is a non zero complex number,
$r = |z|$ and θ the angle from the real axes to the vector z.
Then $x = r\cos\theta \Rightarrow \frac{x}{r} = \cos\theta$
$y = r\sin\theta \Rightarrow \frac{y}{r} = \sin\theta$
so $z = x + iy$ can be written as
$z = r\cos\theta + ir\sin\theta$
$z = r(\cos\theta + i\sin\theta)$ is called the polar form of z
θ is called an argument of z and is denoted by

$$\theta = \arg z$$

but its not unique since we can add or subtract any multiple of 2π to produce another value of the argument,
there is only one value of the argument in radius that satisfies $-\pi < \theta < \pi$
This is called the principle argument of z denoted by $\theta = \operatorname{Arg} z$

Example *(6):*

Exprce the following complex numbers in polar form using the principle arguments
(a) $z = 1 + \sqrt{3}\, i$
(b) $z = -1 - i$

Solution *:*

(a) $r = |z| = \sqrt{4} = 2$
$x = 1, \quad y = \sqrt{3},$
then $1 = 2\cos\theta,$
$\sqrt{3} = 2\sin\theta$
$\cos\theta = \frac{1}{2}, \ \sin\theta = \frac{\sqrt{3}}{2}$
The only value of θ such that $-\pi < \theta \leq \pi$ is $\theta = \frac{\pi}{3} = 60^\circ$
Thus the polar form

$$z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$

(b) $r = |z| = \sqrt{2} \qquad z = -1 - i$
$x = -1, \quad y = -1$
$-1 = \sqrt{2}\cos\theta$
$-1 = \sqrt{2}\sin\theta$
$\cos\theta = \frac{-1}{\sqrt{2}}, \qquad \sin\theta = \frac{-1}{\sqrt{2}}$
The only value of θ that satisfies these relations and $-\pi \leq \theta \leq \pi$
is $\theta = \frac{-3\pi}{4} = -135^\circ$
or $225^\circ = 5\frac{\pi}{4}$
Thus the polar form of z is

$$z = \sqrt[2]{2}\,(\cos\frac{-3}{4}\pi + i\sin\frac{-3}{4}\pi) \qquad -\pi \le \theta \le \pi$$

Since $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

$$z_1 z_2 = r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

since $\cos(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$

hence

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$$

Note that

$$|z_1 z_2| = |z_1||z_2|$$
$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

The product of two complex numbers is obtained by multiplying their moduli and adding their arguments

Now

$$\frac{z_1}{Z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\,\sin(\theta_1 - \theta_2)]$$

where

$$|\tfrac{z_1}{z_2}| = \tfrac{|z_1|}{|z_2|} \qquad \text{if } z_2 \ne 0$$
$$\arg(\tfrac{z_1}{z_2}) = \arg z_1 - \arg z_2$$

Example *(7):*

Let $z_1 = 1 + \sqrt{3}\,i \qquad z_2 = \sqrt{3} + i$

then their polar forms are

$z_1 = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$z_2 = 2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

then $z_1 z_2 = 4[\cos(\frac{\pi}{3} + \frac{\pi}{6}) + i\sin(\frac{\pi}{3} + \frac{\pi}{6})]$

$= 4(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2})$

$= 4(0 + i) = 4i$

$\frac{z_1}{z_2} = 1[\cos(\frac{\pi}{3} - \frac{\pi}{6}) + i\sin(\frac{\pi}{3} - \frac{\pi}{6})]$

$= \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}i$

Remark *:*

$z^n = r^n(\cos n\theta + i\sin n\theta)$

if $r = 1$,then

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

which is called DeMoivres formula.
We now show how DeMoivres formula can be used to obtain roots of complex numbers we have

$$z^{\frac{1}{n}} = \sqrt[n]{r}[\cos(\frac{\theta}{n} + \frac{2k\pi}{n}) + i\sin(\frac{\theta}{n} + \frac{2k\pi}{n})], \quad k = 0, 1, 2, \ldots\ldots, r$$

Example *(8):*

Find all cube roots of -8 that is $\sqrt[3]{-8}$

Solution *:*

$z = -8 \qquad x + iy = -8 \qquad$ hence $x = -8$, $y = 0$

$r = \sqrt{64} = 8$

$\tan\theta = \frac{y}{x} = \frac{0}{-8} = 0$

$\theta = \pi$

so the polar form of -8 is

$-8 = 8(\cos\pi + i\sin\pi)$

then for $n = 3$ if follows that

$(-8)^{\frac{1}{3}} = \sqrt[3]{8}[\cos(\frac{\pi}{3} + \frac{2k\pi}{3}) + i\sin(\frac{\pi}{3} + \frac{2k\pi}{3})]$

for $k = 0, 1, 2,$

Thus the cube roots of -8 are

for $k = 0 \quad = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) = 2(\frac{1}{2} + \frac{\sqrt{3}}{2}i) = 1 + \sqrt{3}\,i$

for $k = 1 \quad = 2(\cos\pi + i\sin\pi) = 2(-1) = -2$

for $k = 2 \quad = 2(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}) = 2(\frac{1}{2} - \frac{\sqrt{3}}{2}i) = 1 - \sqrt{3}\,i$

Find the fourth root of (1) sol.:$(1, i, -1, -i)$

5.3.2:Complex Exponent:

Complex exponents are defind by

$$\cos\theta + i\sin\theta = e^{i\theta}$$

where e is an irrational real number given approximat by
$e \approx 2.71828\ldots$
so $z = r(\cos\theta + i\sin\theta)$,can be written by

$$z = re^{i\theta}$$

5.3.3 Properties of Complex exponents:

if $z_1 = r_1 e^{i\theta_1} \qquad z_2 = r_2 e^{i\theta_2}$,then

1) $z_1 z_2 = r_1 r_2 e^{i(\theta_1 \theta_2)}$

2) $\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

3) $\bar{z} = re^{-i\theta}$

$\bar{z} = r(\cos\theta - i\sin\theta)$

$= r(\cos -\theta + i\sin -\theta)$

$= re^{-i\theta}$

4)If $r = 1$ $z = e^{i\theta}$ and $\overline{e^{i\theta}} = e^{-i\theta}$

Example *(9):*

Find the complex exponent for $Z = 1 + \sqrt{3}\, i$?

Solution *:*

$r = \sqrt{1+3} = 2$, $x = 1 = r\cos\theta \Rightarrow 1 = 2\cos\theta \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$y = \sqrt{3} = r\sin\theta \Rightarrow \sqrt{3} = 2\sin\theta \Rightarrow \sin\theta = \frac{\sqrt{3}}{2}$

$1 + \sqrt{3}\, i = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

$= 2e^{i\frac{\pi}{3}}$.

5.4:Complex vector spaces:

In this section we will develop the basic properties of complex vector spaces.

In complex vector spaces avector w is called alinear combination of the vector $v_1, v_2, \ldots\ldots, v_r$

w can be expresed in the form

$$w = k_1 v_1 + k_2 v_2 + \ldots\ldots + k_r v_r.$$

where $k_1, k_2, \ldots, k_r$ are complex numbers .

linear independence , spanning , basis ,dimensions and subspace carry over without change to complex

vector spaces .

we sea that R^n is the most important vector spaces of n–tuples of real numbers

C^n is the most important vector space of n–tuples of complex numbers with addition and scalar multiplication .

a vector u in C^n can be written in the horizontal as matrix form

$$u = (u_1, u_2, \ldots\ldots, u_n) \quad \text{or} \quad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$$

where $u_1 = a_1 + ib_1$, $u_2 = a_2 + ib_2$, ..., $u_n = a_n + ib_n$.

$e_1 = (1, 0, .., 0),\ e_2 = (0, 1, ..., 0), ..., e_n = (0, 0, ..., 1).$

form abasis called the standrad basis of C^n and since the are $n-$vectors is this basis C^n is an $n-$dimensional

vector space.

If $f_1(x)$ and $f_2(x)$ are real-valued functions of the real variable x, then the expression

$$f(x) = f_1(x) + if_2(x)$$

is called a complex -valued function of the real variable x. some examples are:

$$f(x) = 2x + ix^3 \qquad \text{and} \qquad g(x) = 2\sin x + i\cos x \tag{9.22}$$

Let V be the set of all complex-valued functions defined on the entire line .

If $f = f_1(x) + if_2(x)$ and $g = g_1(x) + ig_2(x)$ are two such functions and k is any complex number,

then we define the sum function $f + g$ and the scalar multiple kf by

$$(f + g)(x) = [f_1(x) + g_1(x)] + i[f_2(x) + g_2(x)]$$
$$(kf) = kf_1(x) + ikf_2(x)$$

In words ,to form $f + g$ add the real parts of f and g and add the imaginary parts .To form kf multiply

the real and imaginary parts of f by k .for example, if $f = f(x)$ and $g = g(x)$ are the functions in(9.22) ,then

$$(f + g)(x) = (2x + 2\sin x) + i(x^3 + \cos x)$$
$$(if)(x) = 2xi + i^2x^3 = -x^3 + 2xi$$

it can be shown that V together with the stated operations is a complex vector space.

Definition *(5.4.1):*

If $u = (u_1, u_2, .., u_n)$ and $v = (v_1, v_2, .., v_n)$ are vectors in C^n then their Euclidean

inner product $u \cdot v$ is defined by

$$u \cdot v = u_1\bar{v}_1 + u_2\bar{v}_2 + ... + u_n\bar{v}_n$$

where $\bar{v}_1, \bar{v}_2, ..., \bar{v}_n$ are the conjugate of $v_1, v_2, ..., v_n$

Example *(10):*

The Euclidean inner product of the vector

$u = (-i, 2, 1 + 3i)$ and $v = (1 - i, 0, 1 + 3i)$

$u \cdot v = (-i)(1 + i) + 2(0) + (1 + 3i)(1 - 3i)$

$= -i + 1 + 1 + 9 = 11 - i.$

Theorem *(5.4.2):*

If $u, v,$ and w are vectors in C^n, and k is any complex number ,then

(a) $u \cdot v = \overline{v \cdot u}.$
(b) $(u + v) \cdot w = u \cdot w + v \cdot w.$
(c) $(ku) \cdot v = k(u \cdot v).$
(d) $v \cdot v \geq 0.$ further , $v \cdot v = 0$ if and only if $v = 0.$

Proof :

(a) let $u = (u_1, u_2, \ldots, u_n)$, $v = (v_1, v_2, \ldots, v_n)$,then

$u \cdot v = u_1\overline{v}_1 + u_2\overline{v}_2 + \ldots\ldots\ldots\ldots + u_n\overline{v}_n$

and

$$v \cdot u = v_1\overline{u}_1 + v_2\overline{u}_2 + \ldots\ldots\ldots + v_n\overline{u}_n.$$

so $\overline{v \cdot u} = \overline{v_1\overline{u}_1 + v_2\overline{u}_2 + \ldots\ldots\ldots\ldots + v_n\overline{u}_n}$

$$= \overline{v_1\overline{u}_1} + \overline{v_2\overline{u}_2} + \ldots\ldots\ldots\ldots\ldots + \overline{v_n\overline{u}_n}$$

$$= \overline{v}_1\overline{\overline{u}_1} + \overline{v}_2\ \overline{\overline{u}_2} + \ldots\ldots\ldots\ldots + \overline{v}_n\ \overline{\overline{u}_n}$$

$$= \overline{v}_1\ u_1 + \overline{v}_2\ \ u_2 + \ldots\ldots\ldots\ldots + \overline{v}_n u_n$$

$$= u_1\overline{v}_1 + u_2\overline{v}_2\ + \ldots\ldots\ldots\ldots + u_n\overline{v}_n$$
$$= u \cdot v.$$

(d) $v \cdot v = v_1\overline{v}_1 + v_2\ \overline{v}_2 + \ldots\ldots\ldots\ldots + v_n\ \overline{v}_n = |v_1|^2 + |v_2|^2 + \ldots\ldots\ldots\ldots + |v_n|^2 \geq 0$

equality holds iff $|v_1| = |v_2| = \ldots\ldots\ldots = |v_n| = 0$

and its true iff $v_1 = v_2 = \ldots\ldots\ldots\ldots\ldots = v_n = 0$ that if iff $v = 0.$

Note that :

$$u \cdot (kv) = \overline{k}(u \cdot v)$$

Definition *(5.4.2):*

The Euclidean norm or (Euclidean length of a vector) $u = (u_1, u_2, \ldots, u_n)$ in C^n is defined by

$$\|u\| = (u \cdot u)^{\frac{1}{2}} = \sqrt{|u_1|^2 + |u_2|^2 + \ldots + |u_n|^2}.$$

Definition *(5.4.3):*

The Euclidean distence betiween the points $u = (u_1, u_2, \ldots, u_n)$ and $v = (v_1, v_2, \ldots, v_n)$ is defined by

$$d(u,v) = \|u - v\| = \sqrt{|u_1 - v_1|^2 + |u_2 - v_2|^2 + \ldots + |u_n - v_n|^2}$$

Example *(11):*

If $u = (i, 1+i, 3)$ and $v = (1-i, 2, 4i)$ then find $\|u\|, d(v,v)$?

Solution *:*

$$\|u\| = \sqrt{|i|^2 + |1+i|^2 + |3|^2} = \sqrt{(i)(-i) + (1+i)(1-i) + 4}$$

$$= \sqrt{1+2+9} = \sqrt{12} = 2\sqrt{3}$$

$$d(u,v) = \sqrt{|-1+2i|^2 + |-1+i|^2 + |3-4i|^2}$$

$$= \sqrt{(-1+2i)(-1-2i) + (-1+i)(-1-i) + (3-4i)(3+4i)}$$

$$= \sqrt{5+2+25} = \sqrt{32} = 4\sqrt{2}.$$

The vector space C^n with norm and inner product is called coplex Euclidean n-space.

Definition *(5.4.4):*

An inner product on a complex vector space V is a function that associates a complex number $\langle u, v\rangle$
with each pair of vector u and v in V such that for all vectors u, v and w in V and all scalars k.

(i) $\langle u, v\rangle = \overline{\langle v, u\rangle}$
(ii) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$
(iii) $\langle ku, v\rangle = k\langle u, v\rangle$
(iiii) $\langle v, v\rangle \geq 0$ and $\langle v, v\rangle = 0$ if and only if $v = 0$

A complex vector space with an inner product is called a complex inner product space or a unitary space .

some more properties:

(i) $\langle 0, v\rangle = \langle v, 0\rangle = 0$
(ii) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$
(iii) $\langle u, kv\rangle = \bar{k}\langle u, v\rangle$

Proof :

$$\langle u, kv\rangle = \overline{\langle kv, u\rangle}$$
$$= \overline{k\langle v, u\rangle}$$
$$= \bar{k}\overline{\langle v, u\rangle}$$
$$= \bar{k}\langle u, v\rangle$$

Example *(12):*

let $U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix}$ and $V = \begin{pmatrix} v_1 & v_2 \\ v_3 & v_4 \end{pmatrix}$

are any 2×2 matries with complex entrie
Now we define the complex inner product on complex $M_{2,2}$
$\langle u, v\rangle = u_1\bar{v}_1 + u_2\bar{v}_2 + \ldots + u_n\bar{v}_n.$

Example *(13) :*

If $u = \begin{pmatrix} 0 & i \\ 1 & 1+i \end{pmatrix}$ and $v = \begin{pmatrix} 1 & -i \\ 0 & 2i \end{pmatrix}$

$$\begin{aligned}\langle u, v\rangle &= 0(\bar{1}) + i(\overline{-i}) + 1(\bar{0}) + (1+i)(\overline{2i}) \\ &= 0 + i^2 + 0 - 2i + 2 \\ &= 1 - 2i\end{aligned}$$

Example *(14) :*

The vectors $u = (i, 1)$ and $v = (1, i)$ in C^2 are orthogonal with respect to the Eulicdeant inner prodcut

$$\langle u, v\rangle = i(\bar{1}) + 1(\bar{i}) = 0$$

Example *(15) :*

Consider the veotor space C^3 with the Euclidean inner prodcut , Apply Gram Schmidt Process
to transform the basis $u_1 = (i, i, i)$, $u_2 = (0, i, i)$ and $u_3 = (0, 0, i)$ in to an orthonormal basis .

Solution *:*

$v_1 = u_1 = (i, i, i)$

$v_2 = u_2 - \frac{\langle u_2, v_1\rangle}{\|v_1\|^2} v_1$

$= (0, i, i) - \frac{2}{3}(i, i, i) = (\frac{-2}{3} i, \frac{1}{3} i, \frac{1}{3} i)$

$v_3 = u_3 - \frac{\langle u_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle u_3, v_2\rangle}{\|v_2\|^2} v_2$

$= (0, 0, i) - \frac{1}{3}(i, i, i) - \frac{\frac{1}{3}}{\frac{2}{3}}(\frac{-2}{3} i, \frac{1}{3} i, \frac{1}{3} i)$
$= (0, \frac{-1}{2} i, \frac{1}{2} i).$
$\|v_1\| = \sqrt{3}$, $\|v_2\| = \frac{\sqrt{6}}{3}$, $\|v_3\| = \frac{1}{\sqrt{2}}$
So , the orthonormal basis are:
$w_1 = \frac{v_1}{\|v_1\|^2} = (\frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}}, \frac{i}{\sqrt{3}})$

$w_2 = \frac{v_2}{\|v_2\|^2} = (\frac{\frac{-2}{3} i, \frac{1}{3} i, \frac{1}{3} i}{\frac{\sqrt{6}}{3}}) = (\frac{-2}{\sqrt{6}} i, \frac{1}{\sqrt{6}} i, \frac{1}{\sqrt{6}} i)$

$$w_3 = \frac{v_3}{\|v_3\|^2} = \frac{(0, \frac{-i}{2}, \frac{i}{2})}{\frac{1}{\sqrt{2}}} = (0, \frac{-i}{\sqrt{2}}, \frac{i}{\sqrt{2}}).$$

5.5: Unitary , Normal , and Hermitian Matrices

Definition *5.5.1 :*

If A is a matrix with complex elements , then the coniugat transpose of A denoted by A^* , is defined by

$$A^* = \overline{A}^t$$

Where $\overline{A}$ is the matrix whose elements are the complex conyugateo of the corresponding entries in A and $\overline{A}^t$ in the transpos of $\overline{A}$.

Example *(16):*

Let $A = \begin{pmatrix} 1+i & -i & 0 \\ 2 & 3-2i & i \end{pmatrix}$ find A^*?

Solution *:*

$$\overline{A} = \begin{pmatrix} 1-i & +i & 0 \\ 2 & 3+2i & -i \end{pmatrix}$$

$$\overline{A}^t = \begin{pmatrix} 1-i & 2 \\ i & 3+2i \\ 0 & -i \end{pmatrix} = A^*$$

Theorem *5.5.2 :*

If A and B are matrices with complex entries and K is any complex number , then :

a) $(A^*)^* = A$.
b) $(A+B)^* = A^* + B^*$
c) $(KA)^* = \overline{K}A^*$
d) $(AB)^* = B^* A^*$

Definition *5.5.3 :*

A square matrix A with complex entries is called unitary matrix if

$$A^{-1} = A^*$$

Theorem *5.5.4:*

If A is an $n \times n$ matrix with complex enteries , then the following are equivelent:

a) A is unitary.

b) The row vector of A form an orthonormal set in C^n with the Euclidean inner prodcut.

c) The column vectors of A form an orthonormal set in C^n with the Euclidean inner prodcut .

Example *(17) :*

The matrix $A = \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1+i}{2} \end{pmatrix}$

has row vectors

$$r_1 = \left(\frac{1+i}{2}, \frac{1+i}{2}\right) \quad , r_2 = \left(\frac{1-i}{2}, \frac{-1+i}{2}\right)$$

Relative to the Euclidean inner product on C^n, we have :

$$\|r_1\| = \sqrt{\left|\frac{1+i}{2}\right|^2 + \left|\frac{1+i}{2}\right|^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

$$\|r_2\| = \sqrt{\left|\frac{1-i}{2}\right|^2 + \left|\frac{-1+i}{2}\right|^2} = \sqrt{\frac{1}{2} + \frac{1}{2}} = 1$$

And

$$r_1 . r_2 = \left(\frac{1+i}{2}\right)\left(\overline{\frac{1-i}{2}}\right) + \left(\frac{1+i}{2}\right)\left(\overline{\frac{-1+i}{2}}\right)$$

$$= \left(\frac{1+i}{2}\right)\left(\frac{1+i}{2}\right) + \left(\frac{1+i}{2}\right)\left(\frac{-1-i}{2}\right) = i - i = 0$$

So, the row vectors from an orthonormal set in C^2 . Thus , A is unitary and

$$A^{-1} = A^* = \begin{bmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1-i}{2} \end{bmatrix}$$

$$AA^* = A^*A = I$$

$$AA^{-1} = A^{-1}A = I$$

Definition *5.5.5 :*

A square matrix A with real entries is called orthogonally diagonollizable if there is a unitary P

such that $P^{-1}AP = (P^*AP)$ is diagonal, the matrix P is said to be unitarily diagonal.

Definition *5.5.6 :*

A square matrix A with complex entries is calld Hemition if

$$A = A^*$$

Example *(18) :*

If

$$A = \begin{pmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{pmatrix}$$

Then

$$\overline{A} = \begin{pmatrix} 1 & -i & 1-i \\ i & -5 & 2+i \\ 1+i & 2-i & 3 \end{pmatrix}$$

So ,

$$A^* = \overline{A}^t = \begin{pmatrix} 1 & i & 1+i \\ -i & -5 & 2-i \\ 1-i & 2+i & 3 \end{pmatrix}$$

Which means that A is Hermition .
In Hermition matrices , the entries on the main diagonal are real numbers .
and mirror image of each entry across the main diagonal is its complex conjugate .
Hermition matrices have some of the properties of real symmetric matrices
Hermition matrices are unitary digonalizable
There are unitarily diagonalizable mstrices that are not Hermition .

Definition *5.5.7 :*

A square matrix A with complex entries is called normal if

$$AA^* = A^*A$$

Note that :

Every Hermition matrix A is normal since

$$AA^* = AA = A^*A$$

And every unitary matrix A is normal since

$$AA^* = I = A^*A$$

Theorem *5.5.8 :*

If A is a square matrix with complex entries , then the following are equavalent :
a) A is unitary diagonalizable .

b) A has an orthonnrmal set of n eigenvectors .

c) A is normal .

Theorem *5.5.9 :*

If A is a normal matrix , then eigenvectors from different eigenspace are orthogonal .

A normal matrix A is diagonalizable by any unitary matrix whose column vectors are eigenvector of A by the following method .

1) Find a basis for each eigenspace of A .

2) Apply Gram – Schmidt proces to each of these basis to optain an orthonormal basis for each eigenspace .

3) From the matrix P whose columns are the basis vector constructed in (2) .

This matrix unitarily diagonalizable A .

Example *(18):*

The matrix $A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$

is unitorily diagonolizable because it is Hermition and therefore normal.

Find a matrix P that unitorily diagonalizes A

Solution *:*

The charactiristic polymonial of A is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -1-i \\ -1+i & \lambda-3 \end{vmatrix} = (\lambda-2)(\lambda-3)-2 = 0$$

$$= (\lambda-1)(\lambda-4) = 0$$

The eigenvalues are $\lambda = 1$ and $\lambda = 4$

be definition

$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is an eigenvector of A corresponding to λ iff

x is a nontrivial solution of

$$\begin{pmatrix} \lambda-2 & -1-i \\ -1+i & \lambda-3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

to find the eigenvector corresponding to $\lambda = 1$

$$\begin{pmatrix} -1 & -1-i \\ -1+i & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = (-1-i)s$, $x_2 = s$

$$x = s\begin{pmatrix} -1-i \\ 1 \end{pmatrix}$$

thus the eigenspace is one dimensional with basis

$$u = \begin{pmatrix} -1-i \\ 1 \end{pmatrix}$$

G.S.P involves only normalizing the vector

$$\|u\| = \sqrt{|-1-i|^2 + |1|^2} = \sqrt{3}$$

$$P_1 = \frac{u}{\|u\|} = \begin{pmatrix} \frac{-1-i}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$$

P_1 is an arthonormal basis for the eigenspace corresponding to $\lambda = 1$
the eigenvector corresponding to $\lambda = 4$

$$\begin{pmatrix} 2 & -1-i \\ -1+i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$x_1 = (\frac{1+i}{2})s \quad , \quad x_2 = s$
so the eigenvectors of A corresponding to $\lambda = 4$ and

$$x = s\begin{pmatrix} \frac{1+i}{2} \\ 1 \end{pmatrix}$$

the eigenspace is one dimentional with basis

$$u = \begin{pmatrix} \frac{1+i}{2} \\ 1 \end{pmatrix}$$

Applying G.S.P.

$$\|u\| = \sqrt{\left|\frac{1+i}{2}\right|^2 + |1|^2} = \sqrt{\frac{3}{2}}$$

$$P_2 = \frac{u}{\|u\|} = \begin{pmatrix} \frac{1+i}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{pmatrix}$$

P_2 is an arthonormal basis for the eigenspace corresponding to $\lambda = 4$

Thus

$$P = [P_1 \vdots P_2] = \begin{pmatrix} \frac{-1-i}{\sqrt{3}} & \frac{1+i}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \end{pmatrix}$$

P diagonalizes A and

$$P^* AP = P^{-1} AP = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Theorem *5.5.1:*

The eigenvalues of a Hermitian matrix and real numbers.

Proof :

If λ is an eigenvalue and v a corresponding eigenvector of an $n \times n$ Hermitian matrix A, then

$$Av = \lambda v$$

Multiplying both sides of this equation on the left by the conjugatc transpose of v yields

$$v^* Av = v^*(\lambda v) = \lambda v^* v \qquad *$$

we will show that the $| \ x \ |$ matrices $v^* Av$ and $v^* v$ both have real entries, so it will follow from (*) that λ must be a real number.

But $v^* Av$ and $v^* v$ are Hermitian, since

$$(v^* Av)^* = v^* A^* (v^*)^* = v^* v$$

and

$$(v^* v)^* = v^* (v^*)^* v = v^* v$$

Since Hermitian matrices have real entries on the main diagonal, and and since $v^* Av$ and $v^* v$ are $| \ x \ |$, it follows that these matrices have real entries, which complet the proof.

Corollary *5.5.11:*

The eigenvalues of a symmetric matrix with real entries are real numbers.

Proof :

Let A be a symmetric matrix with real entries. Because the entries in A are real, it follows that

$$\overline{A} = A$$

But this implies that A is Hermitian, since

$$A^* = (\overline{A})' = A' = A$$

Thus, A has real eigenvalues by Theorem 5.3.10

PROBLEM SET XIII

(1) In each part find the principal argument of z.

(a) $z = -i$ (b) $z = 1 + i$ (c) $z = -1 + \sqrt{3}\, i$

(2) In each part express the complex number in polar form using its principal argument.

(a) $5 + 5i$ (b) $6 + 6\sqrt{3}\, i$ (c) $-3 - 3i$

(3) Express $z_1 = i$, $z_2 = 1 - \sqrt{3}i$, and $z_3 = \sqrt{3} + i$ in polar form and use your results to find

$z_1 z_2 / z_3$. Check your result by performing the calculation without using polar forms.

(4) In each part find all the roots.

(a) $(-i)^{1/2}$ (b) $(-27)^{1/3}$ (c) $\left(-8 + 8\sqrt{3}i\right)^{1/4}$

(5) Find all solutions of the equation.

$Z^{4/3} = -4$

(6) In each part find $Re(z)$ and $Im(z)$.

(a) $z = 3e^{i\pi}$ (b) $z = \sqrt{2}\, e^{\pi 1/2}$ (c) $z = -3e^{-2\pi i}$

(7) Let $u = (2i, 0, -1, 3)$, $v = (-i, i, 1 + i, -1)$, and $w = (1 + i, -i, -1 + 2i, 0)$.

Find

(a) $iv + 2w$ (b) $3\,(u - (1 + i)v)$ (c) $-iv + 2iw$

(8) Let u, v, and w be the vectors in Exercise 8. Find the vector x that satisfies

$u - v + ix = 2ix + w.$

(9) Let $u_1 = (1 - i, i, 0)$, $u_2 = (2i, 1 + i, 1)$ and $u_3 = (0, 2i, 2 - i)$. Find scalars c_1, c_2, and c_3 such that

$c_1 u_1 + c_2 u_2 + c_3 u_3 = (-3 + i, 3 + 2i, 3 - 4i)$

(10) Find the Euclidean norm of v if

(i) $v = (1 + i, 3i, 1)$

(ii) $v = (2i, 0, 2i + 1, -1)$

(11) Let $u = (3i, 0, -i)$, $v = (0, 3 + 4i, -2i)$, and $w = (1 + i, 2i, 0)$. Find.

(i) $\|u\| + \|v\|$

(ii) $\|-iu\| + i\,\|u\|$

(12) Show that if v is a nonzero vector in C^n, then $(1/\|v\|)\,v$ has Euclidean norm 1.

(13) Find the Euclidean inner product $u \cdot v$ if

(i) $u = (-i, 3i)$, $v = (3i, 2i)$

(ii) $u = (1 - i, 1 + i, 2i, 3)$, $v = (4 + 6i, -5i, -1 + i, i)$

(14) Determine which sets are vector spaces under the given operations
For those that are not, list all axioms that fail to hold.

(22) Determine the dimension of and a basis for the solution space of the system.

(i)
$$\begin{aligned} 2x_1 - (1+i)x_2 &= 0 \\ (-1+i)x_1 + x_2 &= 0 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 + ix_2 - 2ix_3 + x_4 &= 0 \\ ix_1 + 3x_2 + 4x_3 - 2ix_4 &= 0 \end{aligned}$$

(23) Prove: If u and v are vectors in complex Euclidean n-space, then

$$u \cdot (kv) = \bar{k}(u \cdot v)$$

(24) Establish the identity.

$$u.\, v = \frac{1}{4}\|u+v\|^2 - \frac{1}{4}\|u-v\|^2 + \frac{i}{4}\|u+iv\|^2 - \frac{i}{4}\|u-iv\|^2$$

for vectors in complex Euclidean n-space.

Problem set XIV

(1) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Show that $\langle u, v\rangle = 3u_1\overline{v_1} + 2u_2\overline{v_2}$ defines an inner product on C^2.

(2) Compute $\langle u, v\rangle$ using the inner product Exercise 1.
(i) $u = (2i, -i)$, $v = (-i, 3i)$.
(ii) $u = (1 + i, 1 - i)$, $v = (1 - i, 1 + i)$.

(3) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Show that
$$\langle u, v\rangle = u_1\overline{v}_1 + (1 + i)u_1\overline{v}_2 + (1 - i)u_2\overline{v}_1 + 3u_2\overline{v}_2$$
defines an inner product on C^2

.(4)Compute $\langle u, v\rangle$ using the inner product in Exercise 3 .
(i) $u = (2i, -i)$, $v = (-i, 3i)$.
(ii) $u = (0, 0)$, $v = (1 - i, 7 - 5i)$.
(iii) $u = (3i, -1 + 2i)$, $v = (3i, -1 + 2i)$

(5) Let $u = (u_1, u_2)$ and $v = (v_1, v_2)$. Determine which of the following are inner products on C^2.
for those that are not , list the axioms that do not hold .
(i) $\langle u, v\rangle = u_1\overline{v}_1 - u_2\overline{v}_2$.
(ii) $\langle u, v\rangle = 2u_1\overline{v}_1 + iu_1\overline{v}_2 + iu_2\overline{v}_1 + 2u_2\overline{v}_2$.

(6) Let $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$. Dose $\langle u, v\rangle = u_1\overline{v}_1 + u_2\overline{v}_2 + u_3\overline{v}_3 - iu_3\overline{v}_1$ define an inner product on C^3? If not ,list all axioms that fail to hold.

Problem set XV

(1) In each part find $A,^*$

(i) $A = \begin{bmatrix} 2i & 1-i \\ 4 & 3+i \\ 5+i & 0 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 7i & 0 & -3i \end{bmatrix}$

(2) Which of the following are Hermitian matrices ?

(i) $\begin{bmatrix} 1 & 1+i \\ 1-i & -3 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 1-i & 1+i \\ 1+i & 0 & 3 \\ -1-i & 3 & 5 \end{bmatrix}$

(3) Find $k, \ell,$ and m to make A a Hermitian matrix .

$$A = \begin{bmatrix} -1 & k & -i \\ 3-5i & 0 & m \\ 1 & 2+4i & 2 \end{bmatrix}$$

(4) Determine which of the following are unitary matrices .

(i) $\begin{bmatrix} \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$ (ii) $\begin{bmatrix} 1+i & 1+i \\ 1-i & -1+i \end{bmatrix}$

(iii) $\begin{bmatrix} \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \\ 0 & \frac{-i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \\ \frac{i}{\sqrt{2}} & \frac{i}{\sqrt{6}} & \frac{i}{\sqrt{3}} \end{bmatrix}$

(5) In each part verify that the matrix is unitary and find its inverse .

(i) $\begin{bmatrix} \frac{3}{5} & \frac{4}{5}i \\ \frac{-4}{5} & \frac{3}{5}i \end{bmatrix}$ (ii) $\begin{bmatrix} \frac{1}{4}(\sqrt{3}+i) & \frac{1}{4}(1-i\sqrt{3}) \\ \frac{1}{4}(1+i\sqrt{3}) & \frac{1}{4}(i-\sqrt{3}) \end{bmatrix}$

(6) Find a unitary matrix P that diagonalizes A , and determine $P^{-1}AP$.

(i) $\begin{bmatrix} 4 & 1-i \\ 1+i & 5 \end{bmatrix}$ (ii) $\begin{bmatrix} 6 & 2+2i \\ 2+2i & 4 \end{bmatrix}$

(iii) $\begin{bmatrix} 2 & \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{-i}{\sqrt{2}} & 2 & 0 \\ \frac{i}{\sqrt{2}} & 0 & 2 \end{bmatrix}$

(7) Prove : If A is invertible , then so is A^* in which case $(A^*)^{-1} = (A^{-1})^*$

(8) Show that if A is unitary matrix, then A^* is also unitary.