Exerciese (I)

1-Prove that if x and y ae both odd ,then $x^2 + y^2$ is even but not divisible by 4.

2-Prove that $5 \mid 8^n - 3^n$ for all non-negative integeral values of n.

3-Show that $3^{2n+1} + 2^{n+2}$ is divisible by 7 for all non-negative integer n.

4-show that if a, b, c and d are integers with a and c non-zero such that $a \mid b$ and $c \mid d$ then $ac \mid bd$.

5-Show that if a, b and $c \neq 0$ are integers then $a \mid b \Leftrightarrow ac \mid bc$.

6-Show that if a and b are integrs such that $a \mid b$ then $a^k \mid b^k$ for every positive integer k.

7-Show that the product of two odd integers is odd ,while the product of two integers is even if either of the integer is even.

8-Prove that the square of any integer of the form $5k+1$ is of the same form , $k \in \mathbf{Z}$.

9-Prove that if an integer is of the form $6k+5$ then it is necessarily of the form $3k-1$,but not conversely, $k \in \mathbf{Z}$.

Exerciese (II)

1-For $n \geq 1$, use induction to show that $3 \mid 2^n + (-1)^{n+1}$.

2-Given that $(a,4) = 2, (b,4) = 2$. Prove that $(a+b,4) = 4$.

3-If $a, b, c \in \mathbf{Z}$, then $(a+kb, b) = (a, b)$.

4-Find (56,72) and express it as a linear combination of 56,72.

5-Find $(5,15), (99,100), (-12,18)$.

6-Let a be a positive integer ,what is the $g.c.d.$ of $a, 2a$?

7-Let a be a positive integer ,what is the $g.c.d.$ of a and $a+1$?

8-Use the Euclidean Algorithm to find the g.c.d. of $(666, 1414)$.

9-Use the Euclidean Algorithm to obtain the integers $x, y \ni (24,138) = 24x + 138y$.

10-Find the g.c.d.of $(70,98,105), (6,10,15)$.

Exerciese (III)

1-Find $[143, 227]$, $[56, 27]$, $[5040, 7700]$.

2-Find the general solution of the Diophantine equation $54x + 21y = 906$.

3-Find all positive solutions of $54x + 21y = 906$.

4-Find the general solution of the following Diophantine equation or show that there is no integral solution and find all positive solutions

(i) $17x + 13y = 100$.
(ii) $60x + 18y = 97$.
(iii) $1402x + 1969y = 1$.
(iv) $25x + 95y = 970$.
(v) $158x - 57y = 7$.

5-Prove that no integers x, y exist satisfying $x + y = 60$ and $(x, y) = 11$.

6-Prove that if a, b are positive integers satisfying $(a, b) = [a, b]$ then $a = b$.

7-Find positive integers a and b satisfying $(a, b) = 10$, and $[a, b] = 100$.

Exerciese (IV)

1-Find the prime factorization of the following:

39, 256, 5040, 9555, 4849845, 210733237.

2-Show that $\sqrt{2} + \sqrt{3}$ is irrational .

3-Using Fermat Factorization method factor each of the following:

143, 2279, 6077, 8051.

4-Find the three smallest even perfect numbers

5-Find a factor of each of the following integers:

$$2^{15} - 1, \quad 2^{1001} - 1.$$

6-Use Theorem 2.4.5, to determinate whether each of the following Mersenne number is prime M_{17}, M_{29}.

7-It has been conjectured that there are infinitly many primes of the form $n^2 - 2$, find five such primes.

8-Prove that the only prime of the form $n^3 - 1$ is 7.

9-Prove that the only primeof the form $8^n + 1. n \geq 1$ is composite.

10-Show that the sum of the twin primes $p, p+2$ is divisible by 12,provided $p > 3$.

11-If $n > 3$ is prime show that $n + 2n + 4$ can't both be primes.

12-Given that p is a prime and $p \mid a^n$, prove that $p^n \mid a^n$.

13-If $(b,c) = 1$ and bc is a perfect square,show that b, c are perfect square.

Exerciese (V)

1-If $a \equiv b (\text{mod}\, m)$, and $d \mid m$, prove that $a \equiv b (\text{mod}\, d)$.

2-If $a \equiv b (\text{mod}\, m)$ and $c > 0$, prove that $ca \equiv cb (\text{mod}\, cm)$.

3-Find the missing number x if $2x99561 = [3(523 + x)^2]$.

4-Solve if possible the following linear congruences :

(i) $18x \equiv 60 (\text{mod}\, 66)$.
(ii) $5x \equiv 11 (\text{mod}\, 29)$.
(iii) $4x \equiv 7 (\text{mod}\, 20)$.

5-Show that each of the following congruences holds

(i) $13x \equiv 1 (\text{mod}\, 2)$.
(i) $91x \equiv 0 (\text{mod}\, 13)$.
(i) $69x \equiv 62 (\text{mod}\, 7)$.

6-Determine whther each of the following pairs of integers are congruent modulo 7,

(a) 1,15 (b) -9,5 (c) 2,99.

7-Show that if a is an even integer ,then $a^2 \equiv 0 \pmod{4}$, and if a is an odd integer ,then
$a^2 1 \pmod{4}$.

8-Use the theory of congruence to verify that $97 \mid 2^{48} - 1$.

9-Without performing the division ,determine whether the integers
176, 521221 and 149,235,678 are divisible by 9 or 11.

10-Show by mathematical induction that if n is a positive integer then
$5^n \equiv 1 + 4n \pmod{16}$.

11-Find all solutions of each of the following linear congruences :

(a) $2x \equiv 5 \pmod{7}$.

(b) $9x \equiv 5 \pmod{25}$.

(c) $103x \equiv 444 \pmod{999}$.

(a) $6x \equiv 3 \pmod{9}$.

12-Find an inverse modulo 17 of each of the following integers: (i) 4 (ii) 7.

13-Find all the solutions of the followig systems of linear congruences:

(a) $x \equiv 1 \pmod{2}$, $x \equiv 2 \pmod{3}$, $x \equiv 3 \pmod{5}$

(b) $x \equiv 5 \pmod{11}$, $x \equiv 14 \pmod{29}$, $x \equiv 15 \pmod{31}$

14-Determine which of the following integers are divisible by 3, and which are divisible by 9.

(a) 18381 (b) 987654321.

15-Which of the following integers is divisible by 11 & 7 & 13.

(a) 10763732 (b) 1086320015.

Exerciese (VI)

1-If $(a, 133) = 1, (b, 133) = 1,$ show that $a^{18} - b^{18} \equiv 0 (\bmod 133)$.

2-Verify that $5^{38} \equiv 4 (\bmod 11)$.

3-If $(n, 7) = 1,$ prove that $7 \mid n^{12} - 1$.

4-If $(n, 13) = 1, (a, 13) = 1$ prove that $13 \mid n^{12} - a^{12}$.

5-If $(n, 91) = 1, (a, 91) = 1$ prove that $n^{12} - a^{12} \equiv 0 (\bmod 91)$.

6-If $(a, 7) = 1,$ prove that $a^{6k} - 1 \equiv 0 (\bmod 7)$ for any $k \in \mathbf{Z}$.
7-Using Wilson's Theorem,find the least positive residue of 8.9.10.11.12.13 modulo 7.
8- Using Fermat's Little theorem ,find the solutions of the following linear congruences
:

(a) $7x \equiv 12 (\bmod 17)$ (b) $7x \equiv 12 (\bmod 17)$.

9-Show that if p is an odd prime then $2(p-3)! \equiv -1 (\bmod p)$.

Exerciese (VII)

1-Find the reduced residues system modulo each of the following integrs :
(a) 6 (b) 14 (c) 10.
2-Use Euler's Theorem to find the least positive residues of $3^{100000} \pmod{35}$.
3-Solve each of the following linear congruences using Euler's Theorem :
(a) $5x \equiv 3 \pmod{14}$ (b) $4x \equiv 7 \pmod{15}$.

4-Find the remaider when
(a) (15)! is divisible by 17 (b) (26)! is divisible by 29.
5- Calculate $\varphi(1001)$, $\varphi(5040)$, $\varphi(254)$.
6-Prove the following:
(i) If n and n+2 are twin primes, then (n+2) = (n)+2.
(ii) If p and 2p+1 are both odd primes, the n = 4p, satisfies
$\varphi(n+2) = \varphi(n) + 2$.

7-Prove that if the integer n has r distinct odd primes factors then $2^r \mid \varphi(n)$.
8-Prove that if (a,p) = 1, proved that $a^{p^{\alpha-1}(\alpha-1)} \equiv 1 \pmod{p^\alpha}, \forall \alpha \in \mathbf{Z}^+$.
9-Use Eulers Thoerem to prove that $a^{37} \equiv a \pmod{1729}$.
10-Show that $\varphi(5186) = \varphi(5187) = \varphi(5188)$.
11-Show that if n is a positive integer ,then

$$\varphi(2n) = \begin{cases} \varphi(n) & if \quad n \text{ is odd} \\ 2\varphi(n) & if \quad n \text{ is even} \end{cases}$$

Exerciese (VII)

1-Verify that (a) $\tau(n) = \tau(n+1) = \tau(n+2) = \tau(n+3)$,

holds for (i) $n = 3655$ (ii) $n = 4503$.

(b) $\sigma(n) = \sigma(n+1)$,

holds for (i) $n = 14$ (ii) $n = 206$ (iii) $n = 957$.

2-Find the sum of the positive integers divisors of each of the following integers :

(a) 35 (b) 196 (c) 10! (d) $2^5.3^4.5^3.7^2.11$.

3-Find the number of positive integers divisors of each of the following integers :

(a) 36 (b) 144 (c) 2.3.5.7.11.13.17.19.

4-Find the value of the Mobius function of each of the following integers :

(a)100 (b) 105 (c) 2.3.5.7.11.13.

5-Prove that if $n > 2$, then $\tau(n) < n$.

6-Prove that if a natural n has precisely three divisors ,the it is the square of prime.

7-Prove that $\sum_{d/n} (\tau(d))^3 = (\sum_{d/n} \tau(d))^2$.

8-Prove that $\sigma(n) = n+1 \Leftrightarrow n$ is a prime.

9Show that 1000! terminates in 249 zeroes.

10-Find the highest power of 5 dividing 1000! and the highest power of 7 dividing 2000!.

Exerciese (IX)

1-Show that 3 is a quadratic residue of 23 but a nonresidue of 19.
2-Use Gauss lemma to evaluate each of the legendere symbols below :

(a) $\left[\frac{8}{15}\right]$ (b) $\left[\frac{5}{19}\right]$ (c) $\left[\frac{6}{31}\right]$.

3-Find all the quadratic residues of each of the following integers :
(i) 5 (ii) 13.

4-Evalute the legendre symbol $\left[\frac{7}{11}\right]$

(a) using Euler's criterion. (b) using Gauss lemma.

5-Find all solutions of the congruance $x^2 \equiv 1 \pmod{15}$.

6-Evaluate each of the following legendre symbol
(a) $\left[\frac{7}{79}\right]$
(b) $\left[\frac{31}{641}\right]$
(c) $\left[\frac{111}{991}\right]$.